ANALYTIC GEOMETRY
CURTISS AND MOULTON

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HEATH

ANALYTIC GEOMETRY

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D. C. HEATH AND COMPANY

BOSTON NEW YORK CHICAGO
ATLANTA SAN FRANCISCO DALLAS
LONDON

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PRINTED IN U.S.A.

PREFACE

The aims and character of the present text are summarized in the following paragraphs. The object here is not so much to present a complete review of the book as to emphasize the purpose of the authors in matters of arrangement and presentation, and to indicate features not to be found in some other texts.

Flexibility. The authors have endeavored to obtain a maximum of flexibility by the inclusion of sufficient material for a year's course so arranged as to be available for shorter courses. A class meeting three hours a week for one semester can cover the ground of the first eight chapters if starred sections are omitted. It is easy to vary this program, either by further omissions or by proceeding more rapidly, so as to include topics presented in later chapters. Superior students should have time for the starred sections, and they will also find abundance of additional material in Chapters IX–XVII.

Accuracy of statement. A textbook in mathematics should not only be written in a clear and simple style but, since one of its main purposes is to teach accuracy of thought, it should be especially distinguished by accuracy of statement. In order that the present volume might conform to these requirements, parts of it have been revised many times. The attempt has been made to phrase every explanation, every set of directions for working exercises, so that there should be no doubt as to what is meant; to use clear-cut definitions carefully stated in advance, and to stick to them; to give complete and satisfactory proofs. The authors are, however, aware that in spite of their efforts some slips are sure to have eluded their vigilance.

Unification. A good technique in elementary algebra and trigonometry is essential to success in the study of analytic geometry. Some texts make a point of employing a minimum

of formulas derived from these sources; for example, they take pains to avoid using the addition formulas for the sine and cosine. In the present volume another plan is followed. An introductory chapter contains formulas and tables from algebra and trigonometry to which frequent reference is subsequently made. Instead of banishing determinants the authors use them in many places, though in the earlier chapters they supply alternative formulas. Reduction and addition formulas for the trigonometric functions are freely employed. In this way the three subjects, algebra, trigonometry, analytic geometry, are presented as a closely-related whole.

The authors have tried to exhibit a similar unity among the various topics of analytic geometry itself. The chapters of this book are tied together, not by means of formal reviews but by cumulative use of material from earlier chapters, especially in the more difficult problems.

Generality of proofs. It is obviously undesirable to give proofs in such a form that they apply only to a particular figure. This has been avoided here. Generality has been secured in some cases by basing proofs on trigonometric formulas which are true for angles of any magnitude.

Fullness of explanation. This text is not of the desiccated sort which promises to provide a short course but fails to do so because its abbreviated explanations must be supplemented by the instructor. Although this book is of about the standard length, the authors hope that they have made their proofs and discussions sufficiently complete. More illustrative examples are here solved than is usual in other analytic geometries. This book is therefore well adapted for home study.

Exercises. The exercises are so arranged that simpler problems of a formal type precede more difficult ones. They are so numbered and subdivided that it is natural to assign a whole exercise as part of a lesson rather than part (*a*) of one exercise, part (*c*) of another, and so on.

The set of answers at the end of the book refers only to odd-numbered problems. A few answers to such problems are omitted when to give them would destroy the usefulness of the exercises.

It will be observed that there are many pairs of problems in which the odd-numbered exercise, whose answer is given at the end of the book, is similar to the even-numbered one whose answer is not given. The instructor may thus follow his preference for one type or the other without omitting essential exercises.

Polar coördinates. Polar coördinates are introduced in the first chapter and their use is developed simultaneously with that of rectangular coördinates in separate divisions placed at the ends of some of the following chapters. It is possible to omit these divisions at first, and to take them up together toward the end of the course. However, the arrangement here adopted represents the authors' preference.

Other features. In order to make close connection with algebra and trigonometry, the authors have devoted Chapter I to elementary curve-plotting such as the student of modern texts in those subjects may have already practised. A fuller treatment, more characteristic of analytic geometry, is given in Chapter VIII. These two chapters are, the authors hope, an improvement on the formidable second chapter of some texts.

Chapter XIII, on curve fitting, contains an explanation of the method of least squares. The treatment is practical throughout and includes instructions for simplifying computation. The use of logarithmic and of semi-logarithmic paper is illustrated. For the statistical material given in the tables the authors are indebted to several of their colleagues, and especially to Professor F. L. Brown of the University of Virginia.

Chapters XIV–XVII present a brief treatment of solid analytic geometry, including what is necessary for the study of certain parts of the calculus.

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PLANE ANALYTIC GEOMETRY

FORMULAS FROM ALGEBRA AND TRIGONOMETRY

1. Quadratic equations. The roots of the quadratic equation

$$Ax^2 + Bx + C = 0 \quad (A \neq 0)$$

are

$$r_1 = \frac{-B + \sqrt{B^2 - 4AC}}{2A}, \quad r_2 = \frac{-B - \sqrt{B^2 - 4AC}}{2A}.$$

When A, B, C are real, *the roots are*

real and unequal if $B^2 - 4AC > 0$,
real and equal if $B^2 - 4AC = 0$,
imaginary if $B^2 - 4AC < 0$.

2. Determinants. The expression

$$\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} \tag{1}$$

is called a **determinant of the second order,** and its value is, by definition, as follows:

$$\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} \equiv a_1b_2 - a_2b_1.$$

The numbers a_1, b_1, a_2, b_2 are the **elements** of the determinant.

A **determinant of the third order** is written and defined as follows:

$$\begin{aligned} \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} &\equiv a_1 \begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} - a_2 \begin{vmatrix} b_1 & c_1 \\ b_3 & c_3 \end{vmatrix} + a_3 \begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix} \\ &\equiv a_1b_2c_3 - a_1b_3c_2 - a_2b_1c_3 + a_2b_3c_1 + a_3b_1c_2 - a_3b_2c_1. \end{aligned} \tag{2}$$

The **minor** of an element is the determinant that remains when the row and column in which the element lies are deleted. The minor of an element of (2) is designated by the corresponding capital letter; for example the minor of a_1 is A_1, the minor of c_2 is C_2, and we have

$$A_1 = \begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix}, \quad C_2 = \begin{vmatrix} a_1 & b_1 \\ a_3 & b_3 \end{vmatrix}.$$

The following identities hold, where D is the determinant in (2):

$$\begin{aligned} D &\equiv a_1A_1 - a_2A_2 + a_3A_3 \\ &\equiv -b_1B_1 + b_2B_2 - b_3B_3 \\ &\equiv c_1C_1 - c_2C_2 + c_3C_3. \end{aligned}$$

Determinants of higher order and their minors are similarly defined. Thus for a determinant of the fourth order we have

$$\begin{vmatrix} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ a_3 & b_3 & c_3 & d_3 \\ a_4 & b_4 & c_4 & d_4 \end{vmatrix} \equiv a_1A_1 - a_2A_2 + a_3A_3 - a_4A_4,$$

where the minor A_1 is the third order determinant remaining when the row and column of a_1 are deleted, and similarly for the other minors A_2, A_3, A_4.

3. Solution of simultaneous linear equations. The solution of two simultaneous linear equations, that is, equations of the first degree,*

---

* We recall the following definitions from algebra: If an equation can be put in the form of a sum of terms equal to zero, where each term is of type ax^my^n (m and n positive integers or zero), then the degree in x, y of a term is the sum of the exponents m and n; the degree in x, y of the equation is the same as that of the term of highest degree.

$$\text{(1)} \qquad \begin{aligned} a_1x + b_1y &= d_1, \\ a_2x + b_2y &= d_2, \end{aligned}$$

can be expressed in determinant notation as follows:

$$\text{(2)} \qquad x = \frac{\begin{vmatrix} d_1 & b_1 \\ d_2 & b_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}}, \quad y = \frac{\begin{vmatrix} a_1 & d_1 \\ a_2 & d_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}}.$$

Similarly, for three simultaneous linear equations

$$\text{(3)} \qquad \begin{aligned} a_1x + b_1y + c_1z &= d_1, \\ a_2x + b_2y + c_2z &= d_2, \\ a_3x + b_3y + c_3z &= d_3, \end{aligned}$$

the solution can be written

$$\text{(4)} \quad x = \frac{\begin{vmatrix} d_1 & b_1 & c_1 \\ d_2 & b_2 & c_2 \\ d_3 & b_3 & c_3 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}}, \quad y = \frac{\begin{vmatrix} a_1 & d_1 & c_1 \\ a_2 & d_2 & c_2 \\ a_3 & d_3 & c_3 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}}, \quad z = \frac{\begin{vmatrix} a_1 & b_1 & d_1 \\ a_2 & b_2 & d_2 \\ a_3 & b_3 & d_3 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}}.$$

Analogous formulas give the solution of n linear equations in n unknowns.

A necessary and sufficient condition that the simultaneous linear equations

$$\begin{aligned} a_1x + b_1y + c_1z &= 0, \\ a_2x + b_2y + c_2z &= 0, \\ a_3x + b_3y + c_3z &= 0, \end{aligned}$$

have a solution other than $x = 0$, $y = 0$, $z = 0$, is that

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = 0$$

4. Logarithms. If a number N is expressed as the xth power of a number a,

$$N = a^x,$$

then the exponent x is called the **logarithm** of N to the **base** a; we write

$$\log_a N = x.$$

Thus by definition

$$a^{\log_a N} = N.$$

Logarithms to the base 10 are called *common logarithms* and are usually written without the subscript 10. As a direct consequence of the definition of a common logarithm, we have

$$\log 1 = 0, \qquad \log 10 = 1.$$

Important properties of logarithms to any base are expressed by the formulas

$$\log_a MN = \log_a M + \log_a N; \qquad \log_a \frac{M}{N} = \log_a M - \log_a N;$$

$$\log_a M^n = n \log_a M; \qquad \log_a \sqrt[r]{M} = \frac{1}{r} \log_a M;$$

$$\log_b N = \frac{\log_a N}{\log_a b}; \qquad \log_a b = \frac{1}{\log_b a}.$$

The common logarithm of N is written as the sum of a positive or negative integer or zero, called the **characteristic,** and a positive decimal called the **mantissa.** Thus we have $\log .023 = -2 + .362$. When the characteristic is negative, as here, we often write it as a positive integer minus 10 or a multiple of 10. Thus $\log .023 = 8.362 - 10$. The mantissa is usually an unending decimal, shortened to three, four, or five digits. To find the characteristic of $\log N$, first find how far it is from the first figure of N that is not a zero to the figure (which may be a zero) just before the decimal point. If the latter figure is k places to the right of the former, the characteristic is k, if k places to the left, the characteristic is $-k$. The mantissa of N is the logarithm of the number obtained from N by moving the decimal point immediately to the left of the first figure of N that is not a zero. Tables of logarithms are tables of mantissas only.

A table of logarithms to three places will be found on page 10.

5. Angles. An angle BAC is generated by rotating a ray (a line extending in but one direction from an end-point) about its end-point A, from the **initial side** AB to the **terminal side** AC. The point A is called the **vertex** of the angle. The rotation may include one or more complete revolutions. We so choose our measurement of angles that a counter-clockwise rotation generates a positive angle, and one in the opposite direction generates a negative angle. Thus in Figure 1 the first and second angles, as indicated

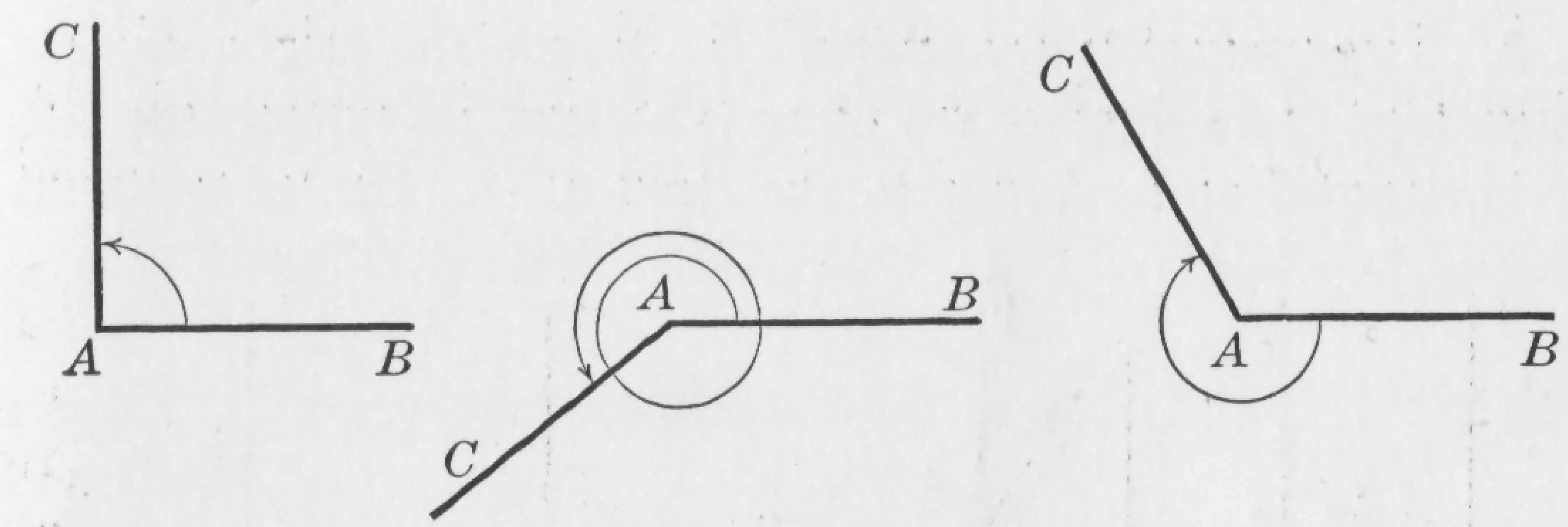

FIG. 1

by the curved arrow, are positive, the second including a complete revolution, while the third is negative.

In one system of measure (degree measure) the unit of measurement is the **degree,** an angle defined as the ninetieth part of a right angle. The sixtieth part of a degree is a *minute*, and the sixtieth part of a minute is a *second*. Thus

$$1 \textit{ right angle} = 90°,$$
$$1° = 60', \quad 1' = 60''.$$

In radian measure the unit angle is the **radian,** an angle such that when its vertex is placed at the center of a circle its sides intercept an arc whose length is equal to that of the radius. By comparing the measures of an angle of half a complete revolution we find

$$180° = \pi \text{ } radians,$$

and from this it follows that

$$1° = \frac{\pi}{180} \; radians = .0174533 \; radians,$$

$$1 \; radian = \frac{180°}{\pi} = 57.29578° = 57°17'45''.$$

If θ is the radian measure of an angle whose vertex is at the center of a circle of radius r, and if s is the length of the intercepted arc, we have

$$s = r\theta.$$

6. Trigonometric functions. To define the trigonometric functions of an angle θ we place it so that its initial side OX is horizontal and extends to the right of O. On its terminal

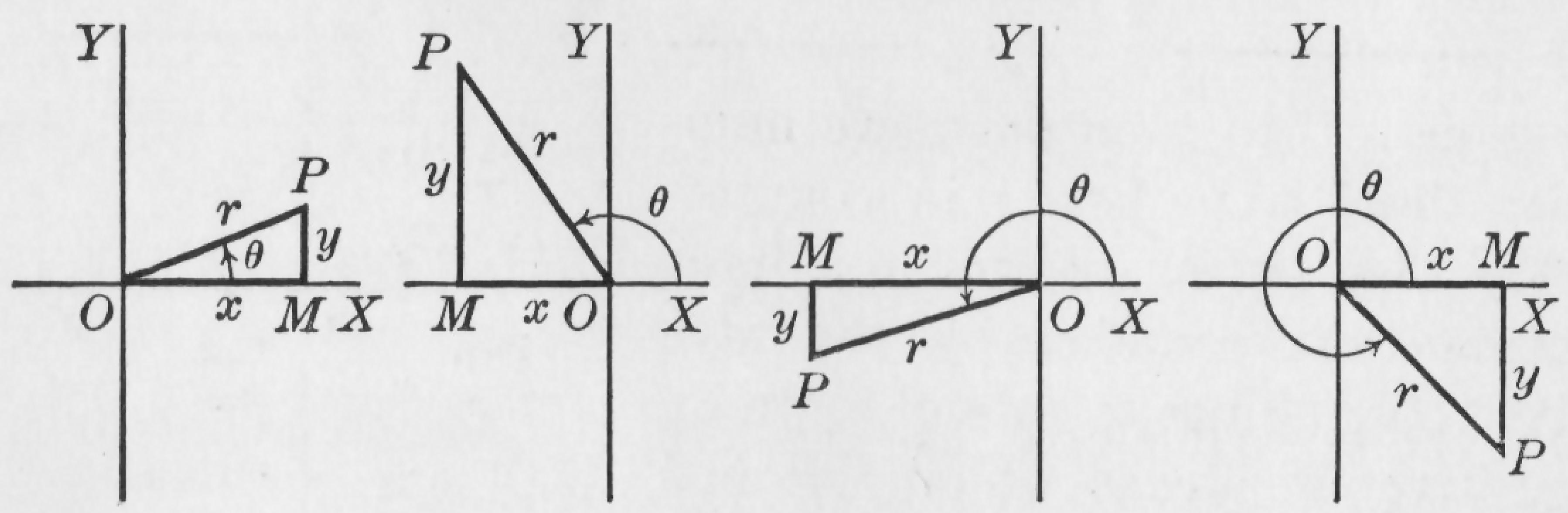

FIG. 2

side we take any point P and drop the perpendicular PM to OX (as in the first and fourth figures above), or to OX produced (as in the second and third figures). In the right triangle OMP, the length of OP is designated by r, and is considered positive. If OM extends to the right (in the

direction of OX), its length is designated x, but if OM extends to the left, x is taken as the negative of its length; similarly y is the length of MP if MP extends upwards (with the same direction as OY in Figure 2), but if the direction from M to P is downwards, y is the negative of the length MP.

With these conventions, we define the trigonometric functions of θ as follows:

$$\begin{aligned}
\text{sine of } \theta &= \sin\theta = \frac{y}{r},\\
\text{cosine of } \theta &= \cos\theta = \frac{x}{r},\\
\text{tangent of } \theta &= \tan\theta = \frac{y}{x},\\
\text{cotangent of } \theta &= \cot\theta = \frac{x}{y},\\
\text{secant of } \theta &= \sec\theta = \frac{r}{x},\\
\text{cosecant of } \theta &= \csc\theta = \frac{r}{y}.
\end{aligned}$$

The lines OX, OY, when produced, divide the plane into four regions called *quadrants;* we number these from 1 to 4 in counterclockwise order, the first quadrant being that bounded by the rays OX, OY. Since x, y, and r are all positive for angles whose terminal sides are in the first quadrant, it follows that all the trigonometric functions of such angles are positive. When OP is in one of the other quadrants, x or y, or both, may be negative. The accompanying diagram shows which of the functions are positive, and which negative, for angles terminating in each of the four quadrants.

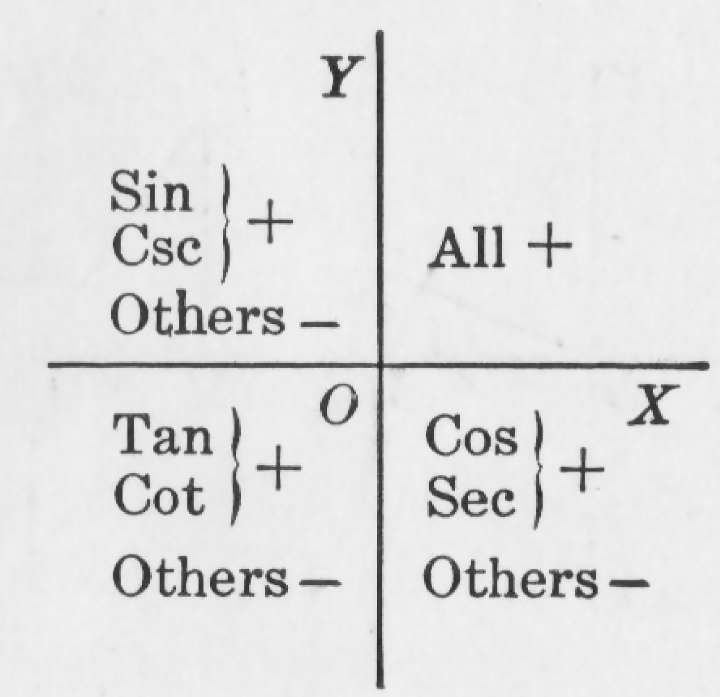

FIG. 3

A table on page 11 gives the radian measures, and the values to three places of the trigonometric functions, of angles from 0° to 90°.

7. Trigonometric identities.

Reduction formulas

$$
\begin{aligned}
\sin(90^\circ - \theta) &= \cos\theta, & \sin(90^\circ + \theta) &= \cos\theta,\\
\cos(90^\circ - \theta) &= \sin\theta, & \cos(90^\circ + \theta) &= -\sin\theta,\\
\tan(90^\circ - \theta) &= \cot\theta, & \tan(90^\circ + \theta) &= -\cot\theta,\\
\sin(180^\circ - \theta) &= \sin\theta, & \sin(180^\circ + \theta) &= -\sin\theta,\\
\cos(180^\circ - \theta) &= -\cos\theta, & \cos(180^\circ + \theta) &= -\cos\theta,\\
\tan(180^\circ - \theta) &= -\tan\theta, & \tan(180^\circ + \theta) &= \tan\theta,\\
\sin(-\theta) &= -\sin\theta, & \sin(360^\circ + \theta) &= \sin\theta,\\
\cos(-\theta) &= \cos\theta, & \cos(360^\circ + \theta) &= \cos\theta,\\
\tan(-\theta) &= -\tan\theta, & \tan(360^\circ + \theta) &= \tan\theta.
\end{aligned}
$$

Formulas involving one angle

$$\cot\theta = \frac{1}{\tan\theta}, \quad \sec\theta = \frac{1}{\cos\theta}, \quad \csc\theta = \frac{1}{\sin\theta},$$

$$\tan\theta = \frac{\sin\theta}{\cos\theta}, \quad \cot\theta = \frac{\cos\theta}{\sin\theta},$$

$$
\begin{aligned}
\sin^2\theta + \cos^2\theta &= 1,\\
1 + \tan^2\theta &= \sec^2\theta,\\
1 + \cot^2\theta &= \csc^2\theta.
\end{aligned}
$$

Addition formulas

$$
\begin{aligned}
\sin(\alpha + \beta) &= \sin\alpha\cos\beta + \cos\alpha\sin\beta,\\
\sin(\alpha - \beta) &= \sin\alpha\cos\beta - \cos\alpha\sin\beta,\\
\cos(\alpha + \beta) &= \cos\alpha\cos\beta - \sin\alpha\sin\beta,\\
\cos(\alpha - \beta) &= \cos\alpha\cos\beta + \sin\alpha\sin\beta,
\end{aligned}
$$

$$\tan(\alpha + \beta) = \frac{\tan\alpha + \tan\beta}{1 - \tan\alpha\tan\beta},$$

$$\tan(\alpha - \beta) = \frac{\tan\alpha - \tan\beta}{1 + \tan\alpha\tan\beta}.$$

Formulas for the double angle and for the half angle

$$
\begin{aligned}
\sin 2\alpha &= 2\sin\alpha\cos\alpha, & \sin\frac{\alpha}{2} &= \pm\sqrt{\frac{1-\cos\alpha}{2}},\\
\cos 2\alpha &= \cos^2\alpha - \sin^2\alpha, & \cos\frac{\alpha}{2} &= \pm\sqrt{\frac{1+\cos\alpha}{2}},\\
&= 2\cos^2\alpha - 1, & \tan\frac{\alpha}{2} &= \pm\sqrt{\frac{1-\cos\alpha}{1+\cos\alpha}},\\
&= 1 - 2\sin^2\alpha, & &= \frac{1-\cos\alpha}{\sin\alpha},\\
\tan 2\alpha &= \frac{2\tan\alpha}{1-\tan^2\alpha}, & &= \frac{\sin\alpha}{1+\cos\alpha}.
\end{aligned}
$$

Formulas for triangles whose sides are a, b, c, and whose opposite angles are A, B, C

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}, \qquad \textit{Law of sines.}$$

$$a^2 = b^2 + c^2 - 2bc\cos A, \qquad \textit{Law of cosines.}$$

$$\textit{Area} = \tfrac{1}{2}bc\sin A.$$

8. The Greek alphabet.

Letters	Names	Letters	Names	Letters	Names
Α α	alpha	Ι ι	iota	Ρ ρ	rho
Β β	beta	Κ κ	kappa	Σ σ ς	sigma
Γ γ	gamma	Λ λ	lambda	Τ τ	tau
Δ δ	delta	Μ μ	mu	Υ υ	upsilon
Ε ε	epsilon	Ν ν	nu	Φ φ	phi
Ζ ζ	zeta	Ξ ξ	xi	Χ χ	chi
Η η	eta	Ο ο	omicron	Ψ ψ	psi
Θ θ	theta	Π π	pi	Ω ω	omega

9. Tables.

I. Three-place common logarithms of numbers

N	0	1	2	3	4	5	6	7	8	9
1	000	041	079	114	146	176	204	230	255	279
2	301	322	342	362	380	398	415	431	447	462
3	477	491	505	519	531	544	556	568	580	591
4	602	613	623	633	643	653	663	672	681	690
5	699	708	716	724	732	740	748	756	763	771
6	778	785	792	799	806	813	820	826	833	839
7	845	851	857	863	869	875	881	886	892	898
8	903	908	914	919	924	929	934	940	944	949
9	954	959	964	968	973	978	982	987	991	996
10	000	004	009	013	017	021	025	029	033	037
11	041	045	049	053	057	061	064	068	072	076
12	079	083	086	090	093	097	100	104	107	111
13	114	117	121	124	127	130	134	137	140	143
14	146	149	152	155	158	161	164	167	170	173
15	176	179	182	185	188	190	193	196	199	201
16	204	207	210	212	215	217	220	223	225	228
17	230	233	236	238	241	243	246	248	250	253
18	255	258	260	262	265	267	270	272	274	276
19	279	281	283	286	288	290	292	294	297	299

II. Square roots of numbers to three places

N	0	1	2	3	4	5	6	7	8	9
0	0.00	1.00	1.41	1.73	2.00	2.24	2.45	2.65	2.83	3.00
1	3.16	3.32	3.46	3.61	3.74	3.87	4.00	4.12	4.24	4.36
2	4.47	4.58	4.69	4.80	4.90	5.00	5.10	5.20	5.29	5.39
3	5.48	5.57	5.66	5.74	5.83	5.92	6.00	6.08	6.16	6.24
4	6.32	6.40	6.48	6.56	6.63	6.71	6.78	6.86	6.93	7.00
5	7.07	7.14	7.21	7.28	7.35	7.42	7.48	7.55	7.62	7.68
6	7.75	7.81	7.87	7.94	8.00	8.06	8.12	8.19	8.25	8.31
7	8.37	8.43	8.49	8.54	8.60	8.66	8.72	8.77	8.83	8.89
8	8.94	9.00	9.06	9.11	9.17	9.22	9.27	9.33	9.38	9.43
9	9.49	9.54	9.59	9.64	9.70	9.75	9.80	9.85	9.90	9.95

III. Natural values of trigonometric functions

Degrees	Radians	Sin	Cos	Tan	Cot	Sec	Csc		
0°	**.000**	.000	1.000	.000	——	1.000	——	**1.571**	**90°**
1°	**.017**	.017	1.000	.017	57.3	1.000	57.3	**1.553**	**89°**
2°	**.035**	.035	.999	.035	28.6	1.001	28.6	**1.536**	**88°**
3°	**.052**	.052	.999	.052	19.1	1.001	19.1	**1.518**	**87°**
4°	**.070**	.070	.998	.070	14.3	1.002	14.3	**1.501**	**86°**
5°	**.087**	.087	.996	.087	11.4	1.004	11.5	**1.484**	**85°**
10°	**.175**	.174	.985	.176	5.67	1.02	5.76	**1.396**	**80°**
15°	**.262**	.259	.966	.268	3.73	1.04	3.86	**1.309**	**75°**
20°	**.349**	.342	.940	.364	2.75	1.06	2.92	**1.222**	**70°**
25°	**.436**	.423	.906	.466	2.14	1.10	2.37	**1.134**	**65°**
30°	**.524**	.500	.866	.577	1.73	1.15	2.00	**1.047**	**60°**
35°	**.611**	.574	.819	.700	1.43	1.22	1.74	**.960**	**55°**
40°	**.698**	.643	.766	.839	1.19	1.31	1.56	**.873**	**50°**
45°	**.785**	.707	.707	1.000	1.00	1.41	1.41	**.785**	**45°**
		Cos	Sin	Cot	Tan	Csc	Sec	Radians	Degrees

IV. Exact natural values of trigonometric functions

Angle	Radians	Sin	Cos	Tan	Cot	Sec	Csc
0°	0	0	1	0	——	1	——
30°	$\frac{1}{6}\pi$	$\frac{1}{2}$	$\frac{1}{2}\sqrt{3}$	$\frac{1}{3}\sqrt{3}$	$\sqrt{3}$	$\frac{2}{3}\sqrt{3}$	2
45°	$\frac{1}{4}\pi$	$\frac{1}{2}\sqrt{2}$	$\frac{1}{2}\sqrt{2}$	1	1	$\sqrt{2}$	$\sqrt{2}$
60°	$\frac{1}{3}\pi$	$\frac{1}{2}\sqrt{3}$	$\frac{1}{2}$	$\sqrt{3}$	$\frac{1}{3}\sqrt{3}$	2	$\frac{2}{3}\sqrt{3}$
90°	$\frac{1}{2}\pi$	1	0	——	0	——	1
120°	$\frac{2}{3}\pi$	$\frac{1}{2}\sqrt{3}$	$-\frac{1}{2}$	$-\sqrt{3}$	$-\frac{1}{3}\sqrt{3}$	-2	$\frac{2}{3}\sqrt{3}$
135°	$\frac{3}{4}\pi$	$\frac{1}{2}\sqrt{2}$	$-\frac{1}{2}\sqrt{2}$	-1	-1	$-\sqrt{2}$	$\sqrt{2}$
150°	$\frac{5}{6}\pi$	$\frac{1}{2}$	$-\frac{1}{2}\sqrt{3}$	$-\frac{1}{3}\sqrt{3}$	$-\sqrt{3}$	$-\frac{2}{3}\sqrt{3}$	2
180°	π	0	-1	0	——	-1	——
270°	$\frac{3}{2}\pi$	-1	0	——	0	——	-1
360°	2π	0	1	0	——	1	——

CHAPTER I

COÖRDINATES AND GRAPHS

Geometry and algebra developed historically as two separate divisions of mathematics. The ancient Greeks had discovered, by the time of Euclid (300 B.C.), practically all of the propositions now taught in high school geometry. They accomplished this, however, with almost no use of algebraic methods. The origins of algebra, on the other hand, may be traced to the Hindus, and after them, to the Arabs of the earlier Middle Ages.

The revival of learning which inaugurated the Modern Age in Western Europe influenced mathematical study profoundly. Advances of especial importance were made in algebra, and the time came when the traditional wall between that subject and geometry was broken down. The first treatise in which this was done systematically, and which was thus the first Analytic Geometry, was the "Geometry" of René Descartes, published in 1637.

The union of geometry and algebra in analytic geometry made possible the development of modern mathematics, and thus of all the other sciences to which mathematics is indispensable. This union was brought about by the use of coördinates, which, as we shall see, enable us to describe a *point*, a fundamental element of geometry, by a pair of *numbers*, objects of study in algebra.

The present chapter contains a review of certain topics regarding coördinates and their applications, which are presented in college algebras and trigonometries. In later chapters, particularly in Chapter VIII, the subjects here introduced will be more completely discussed.

1. Rectangular coördinates. Rectangular coördinates enable us to locate a point P in a plane in terms of the distances of that point from two mutually perpendicular lines, called **coördinate axes.** These two axes, OX, the ***x*-axis,** and OY, the ***y*-axis,** intersect at a point O called the **origin.** We shall consider OX as horizontal, and OY as vertical. From P we drop the perpendicular PM to the x-axis, and the perpendicular PN to the y-axis. Choosing a suitable

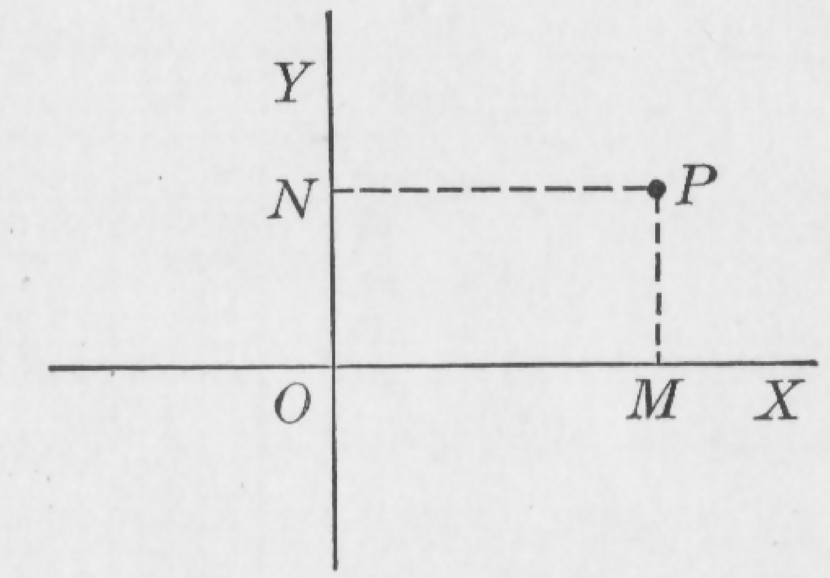

FIG. 4

unit of measurement for lengths on the axes,* we define the **rectangular coördinates** (x, y) of P in terms of the lengths $\overline{OM}$, $\overline{ON}$, as follows:

$$x = \overline{OM} \textit{ if } P \textit{ is to right of } OY,$$
$$x = -\overline{OM} \textit{ if } P \textit{ is to the left of } OY,$$
$$y = \overline{ON} \textit{ if } P \textit{ is above } OX,$$
$$y = -\overline{ON} \textit{ if } P \textit{ is below } OX.$$

If P is on the x-axis, we have $y = 0$, and if on the y-axis, then $x = 0$. We refer to x as the ***x*-coördinate,** or **abscissa,** of P, and to y as the ***y*-coördinate,** or **ordinate,** of P.

When coördinate axes and the unit of measurement have been chosen, each point of the plane has one and only one pair of coördinates, and but one point corresponds to a given pair of coördinates. Thus, by the interpretation of pairs of numbers as coördinates, every pair of real numbers is represented by a point. Imaginary numbers, however, are not represented by points in this way.

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* Although it is possible to use different units for the two axes, we shall not do so in this book, since many of the formulas of analytic geometry require that the units be the same.

2. Plotting a point. Paper cross-ruled horizontally and vertically, as shown in Figure 5, is useful in measuring the coördinates of a given point, and in plotting (that is, locating and marking) a point whose coördinates are given.

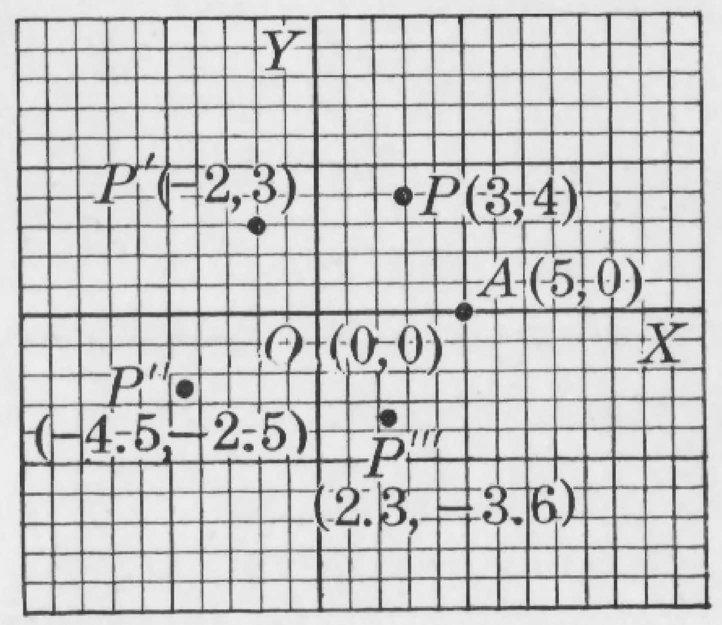


FIG. 5

In Figure 5, if the distance between consecutive rulings is the unit,

the coördinates of P are $x = 3, y = 4$,
the coördinates of P' are $x = -2, y = 3$,
the coördinates of P'' are $x = -4.5, y = -2.5$,
the coördinates of P''' are $x = 2.3, y = -3.6$,
the coördinates of A are $x = 5, y = 0$,
the coördinates of O are $x = 0, y = 0$.

A customary notation places the coördinates in a parenthesis, with the abscissa first; in this way the points which we have plotted are more briefly designated as follows: $P(3, 4)$, $P'(-2, 3)$, $P''(-4.5, -2.5)$, $P'''(2.3, -3.6)$, $A(5, 0)$, $O(0, 0)$.

In order to plot a point P whose coördinates are given, proceed to the right or left from O on OX as many units as indicated by the abscissa, then go up or down as many units as indicated by the ordinate and mark the point thus reached. For example, to plot $(-2, 3)$ we go from O two units to the left, then three up. Clearly we would have reached the same point if we had first gone three units up from O, then two units horizontally to the left.

EXERCISES

Use cross-ruled paper. For the first six Exercises indicate in the figures (as in Figure 5) the coördinates of each point plotted.

1. Choose as unit the distance between consecutive rulings; then plot the following points: $(2, 4)$, $(-3, 1)$, $(-5, -2)$, $(1\frac{1}{2}, -3\frac{1}{2})$, $(-2.5, 0.5)$, $(-2, 0)$.

2. Using the same unit as in Exercise 1, plot the following points: $(3, 1)$, $(-2, 5)$, $(-1, 4)$, $(2\frac{1}{2}, -2\frac{1}{2})$, $(-1.5, 4.5)$, $(4, 0)$.

3. Use twice the distance between consecutive rulings as unit, and plot the following points: $(2, 3.5)$, $(4, -2\frac{1}{2})$, $(-\frac{3}{4}, -4\frac{1}{4})$, $(3.3, -4.6)$, $(0, 5.5)$.

4. Use five times the distance between consecutive rulings as unit, and plot the following points: $(1, 2)$, $(\frac{2}{5}, 1\frac{1}{5})$, $(-1\frac{1}{3}, \frac{2}{3})$, $(-0.2, -1.3)$, $(-1.3, 0)$.

5. A rectangle has its center at the origin and one vertex at the point $(-1, 2)$. If two sides are parallel to the x-axis, find the coördinates of the other vertices. Plot the vertices.

6. A square whose side is of length 2 has one vertex at the origin, another on the upper y-axis, and another on the x-axis to the left of the origin. Find the coördinates of the vertices and plot them.

7. In what quadrant (see page 7) does a point lie:

(*a*) If both coördinates are positive? I
(*b*) If both coördinates are negative? III
(*c*) If the abscissa is positive and the ordinate negative? IV
(*d*) If the abscissa is negative and the ordinate positive? II

8. On what line do all points lie for which $x = 0$? What is the value of the ordinate of every point on the x-axis?

9. What is the locus of points (*a*) for which $x = -5$? (*b*) For which $y = 1$?

10. What is the locus of points for which $y = x$?

11. What equation holds between the x- and y-coördinates of every point on the line that bisects the angle between OX and the negative y-axis? $X = -y$

12. What are the coördinates of a point on the line joining $(0, 0)$ and (a, b), situated half-way between those points? $\left(\frac{a}{2}, \frac{b}{2}\right)$

13. Find the fourth vertex of a parallelogram if three of the vertices are $(0, 0)$, $(a, 0)$, $(0, b)$ (three solutions).

3. Graphs of equations in rectangular coördinates. By a **solution** of an equation in x and y, we mean a **pair of numbers,** a value for x and a value for y, which satisfy the equation. Thus the equation

$$(1) \qquad 2x - y = 3$$

has the solution (2, 1), since if we substitute $x = 2$, $y = 1$, we have $2 \cdot 2 - 1 = 3$; other solutions are (3, 3) and (4, 5).

An equation in x and y has, in general, infinitely many solutions. Thus, no matter what number x_1 may be, a solution of equation (1) is given by $x = x_1$, $y = 2x_1 - 3$, since

$$2x_1 - (2x_1 - 3) = 3.$$

The quantities x and y vary as we pass from one solution to another, hence they are called **variables.** Fixed numbers, such as the numbers 2 and 3 in equation (1), are called **constants.**

We obtain a geometric representation of a solution (x, y) of an equation by plotting the point (x, y). The points which thus represent solutions of an equation form a locus which is defined more explicitly as follows:

The locus, in rectangular coördinates, of an equation in two variables, x *and* y, *is the locus of all points whose coördinates* (x, y) *satisfy the equation.*

We obtain a picture of such a locus by plotting a number of its points and connecting them by a smooth curve. This curve is called the **graph** of the equation.

To draw the graph of an equation we must first obtain a number of solutions. This may be done by solving the equation for y in terms of x, substituting a number of values for x, and computing the corresponding values of y. In some cases it is more convenient to solve for x in terms of y.

For example, if we solve equation (1) for y we have

$$y = 2x - 3.$$

We see that

$$\text{if } x = 0, \text{ then } y = 2 \cdot 0 - 3 = -3,$$
$$\text{if } x = 1, \text{ then } y = 2 \cdot 1 - 3 = -1,$$

and we may similarly compute other solutions. The following table presents a convenient arrangement of results:

x	-4	-3	-2	-1	0	1	1.5	2	4
y	-11	-9	-7	-5	-3	-1	0	1	5

When the points $(-4, -11)$, $(-3, -9)$, and the others given by this table are plotted, it is seen that they all lie on the straight line shown in Figure 6. This line is the graph of the equation $2x - y = 3$.

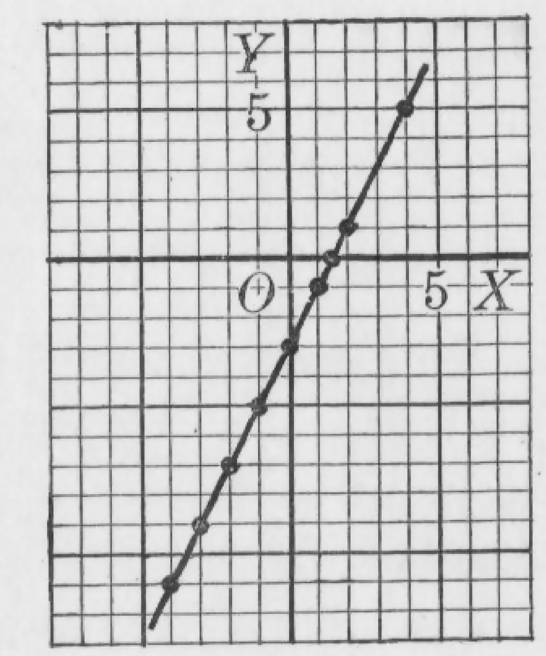


FIG. 6

Example 1. — Draw the graph of the equation $y^2 = 1 - x$.

Solution. — Solve for x in terms of y; this gives

$$x = 1 - y^2.$$

Let y have the values shown in the table below, and compute the corresponding values of x.

x	-15	-8	-3	0	$\frac{3}{4}$
y	-4	-3	-2	-1	$-\frac{1}{2}$

x	1	$\frac{3}{4}$	0	-3	-8	-15
y	0	$\frac{1}{2}$	1	2	3	4

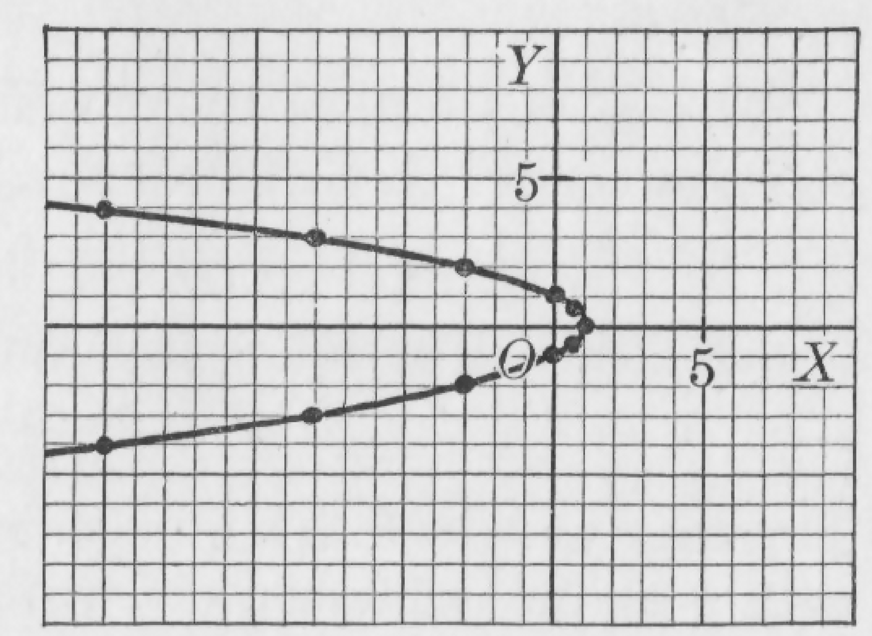


FIG. 7

The graph is given by Figure 7; of course it cannot *all* be shown since a complete graph would extend indefinitely far to the left. Note the advisability of using one or more values of y between -1 and 0, and

between 0 and 1, in order to get a better estimate of the shape of the corresponding part of the curve. This curve is called a **parabola.**

Example 2. — Draw the graph of the equation $9x^2 + 16y^2 = 144$.

Solution. — Solving for y we have

$$16y^2 = 144 - 9x^2,$$
$$y^2 = \tfrac{9}{16}(16 - x^2),$$
$$y = \pm \tfrac{3}{4}\sqrt{16 - x^2}.$$

The following is a table of values of x and y to one place of decimals.*

(a) From $y = +\frac{3}{4}\sqrt{16 - x^2}$	x	−4	−3.5	−3	−2	−1	0	1	2	3	4
	y	0	1.5	2.0	2.6	2.9	3	2.9	2.6	2.0	0
(b) From $y = -\frac{3}{4}\sqrt{16 - x^2}$	x	−4	−3.5	−3	−2	−1	0	1	2	3	4
	y	0	−1.5	−2.0	−2.6	−2.9	−3	−2.9	−2.6	−2.0	0

In Figure 8 the unit is twice the distance between consecutive cross-rulings. The upper half of the graph belongs to (a) and the lower half to (b); the whole is the graph of the given equation. The form of the equations in (a) and (b) shows that x cannot be greater than 4, or less, algebraically, than − 4, if y is to have a real value. Similarly y must be between − 3 and + 3. The curve is symmetrical to both the axes and to the origin. It is called an **ellipse.**

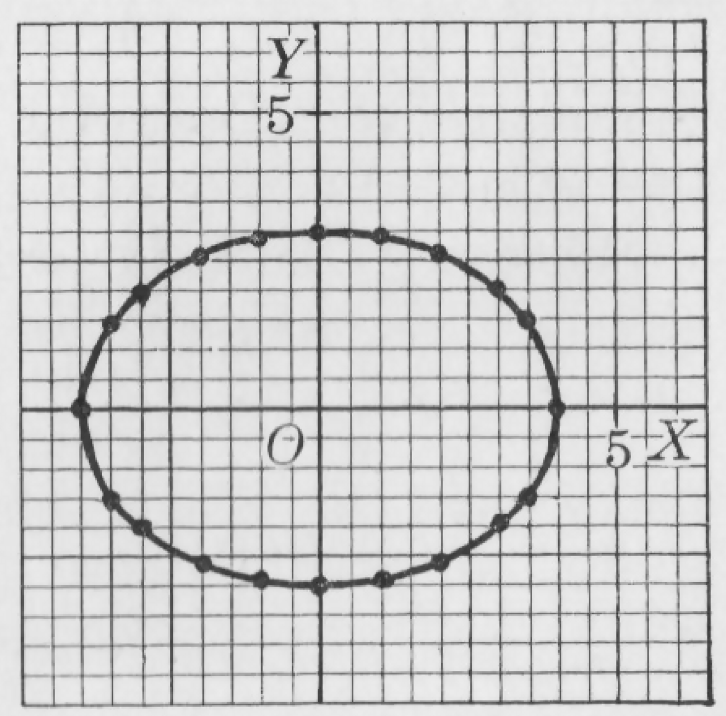


FIG. 8

Example 3. — Draw the graph of the equation

$$x^2 - y^2 + 2x + 2y + 1 = 0.$$

Solution. — The equation may be written

$$y^2 - 2y - (x + 1)^2 = 0.$$

This may be solved as a quadratic in y; we use the formula of page 1, with y as the unknown, and with $A = 1$, $B = -2$, $C = -(x + 1)^2$. The result is

$$y = \frac{2 \pm \sqrt{4 + 4(x + 1)^2}}{2},$$

* The table of square roots on page 10 may be used to compute y.

or

$$(a)\quad y = 1 + \sqrt{1 + (x + 1)^2},$$
$$(b)\quad y = 1 - \sqrt{1 + (x + 1)^2}.$$

The following is a table of solutions:

From (a)	x	− 4	− 3	− 2	− 1	0	1	2	3
	y	4.2	3.2	2.4	2	2.4	3.2	4.2	5.1
From (b)	x	− 4	− 3	− 2	− 1	0	1	2	3
	y	− 2.2	− 1.2	− .4	0	− .4	− 1.2	− 2.2	− 3.1

The upper part of Figure 9 gives the graph of (a), the lower part the graph of (b), the unit being twice the distance between consecutive cross-rulings. These two loci are said to be *branches* of the locus of the given equation. Each branch extends indefinitely far both to the right and to the left. The whole curve is called a **hyperbola.**

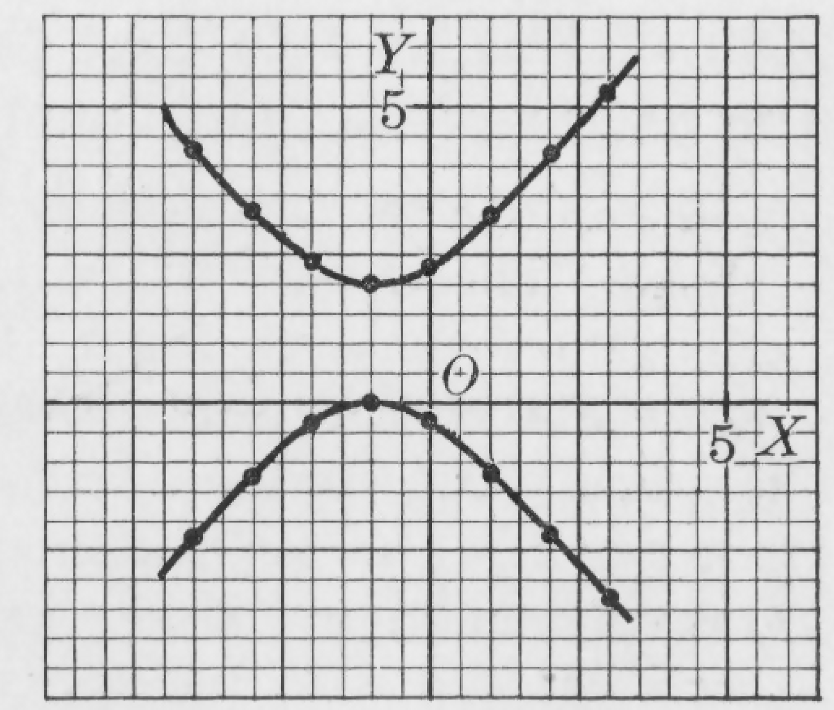


FIG. 9

4. Analytic geometry. Plane analytic geometry is built upon the correspondence between points and the pairs of numbers (x, y) which are their coördinates. By means of this correspondence geometric problems are reducible to algebraic ones, and algebraic formulas and operations may be given geometric interpretations.

Two fundamental problems of analytic geometry are the following:

I. Given a locus defined geometrically, to find a corresponding equation.

II. Given an equation, to find a corresponding locus and its properties.

We have just been considering the problem of drawing a curve which corresponds to a given equation. This is a part of Problem II. It is not a satisfactory solution merely to plot a few points and sketch what seems most likely to be the shape of the curve. In the following chapters we deduce from the equations of various types of curves so much information as to their shape and properties that guesswork is eliminated.

We shall see in Chapters XIV–XVII how analytic geometry is extended from the plane to space by assigning three coördinates to each point, and how equations in three variables are interpreted as loci in space. It is by a generalization of this idea to four or more variables that we obtain the notion of n-dimensional analytic geometry.

EXERCISES

Draw graphs of the following equations.

1. $x + y = 5.$ **2.** $x - y = 5.$

3. $y = 2x + 4.$ **4.** $x = 2y - 4.$

5. $\frac{x}{2} + \frac{y}{3} = 1.$ **6.** $\frac{x}{2} - \frac{y}{3} = 1.$

7. $x = 0.$ **8.** $y = 0.$

9. $2x + 3 = 0.$ **10.** $2y + 3 = 0$

11. $4y = x^2.$ **12.** $4x = y^2.$

13. $y = 2x - x^2.$ **14.** $x = 4y - y^2.$

15. $x^2 + y^2 = 25.$ **16.** $x^2 + y^2 = 169.$

17. $x^2 + 4y^2 = 64.$ **18.** $2x^2 + y^2 = 50.$

19. $x^2 - 4y^2 = 64.$ **20.** $2x^2 - y^2 = 50.$

Tues

21. $xy = 10.$ **22.** $2xy = -15.$

23. $x^2 + y^2 = 6x.$ **24.** $x^2 + y^2 - 4x + 2y - 20 = 0.$

25. $16x^2 + 9y^2 - 32x + 18y = 119.$ **26.** $9x^2 + 25y^2 - 18x + 50y = 191.$

27. $xy^2 = 9.$ **28.** $x^2y + y = 10.$

29. $y = x^3.$ **30.** $y^2 = x^3.$

5. Simultaneous equations. In algebra two equations in x and y are called **simultaneous** when it is required to find one or more pairs of values (x, y) that satisfy both equations. To obtain a geometric interpretation of the problem of solving two simultaneous equations we draw graphs of the equations referred to the same coördinate axes. A pair of real values (x, y) that satisfies both equations corresponds to a point that is on both graphs. Hence *all real * solutions* (x, y) *of two simultaneous equations in* x *and* y *are the coördinates of points of intersection of their graphs; and the coördinates of all points of intersection are solutions.*

Example. — Solve the simultaneous equations

$$x^2 + y^2 - 2x = 24,$$
$$x - y = 4,$$

by finding the points of intersection of their graphs. Check by solving algebraically.

Solution. — In Figure 10 the graphs are shown, the unit being twice the distance between consecutive cross-rulings. The coördinates of the points of intersection are approximately $P(5.7,\ 1.7)$, $P'(-.7,\ -4.7)$. Hence solutions are, to one decimal place, $x = 5.7$, $y = 1.7$, and $x = -.7$, $y = -4.7$.

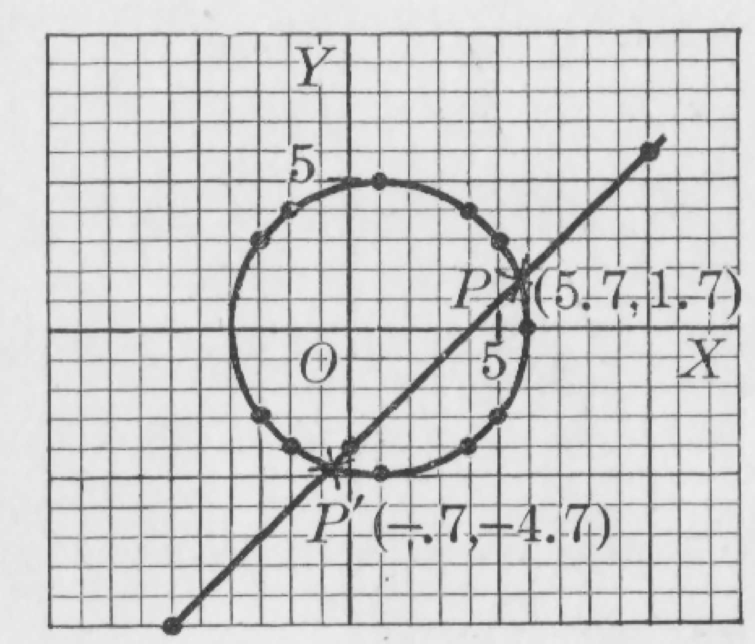


FIG. 10

An algebraic solution is obtained by solving the second equation for y and substituting the result in the first. We thus have

$$y = x - 4,$$
$$x^2 + (x - 4)^2 - 2x = 24,$$
$$2x^2 - 10x - 8 = 0,$$
$$x^2 - 5x - 4 = 0,$$
$$x = \frac{5 \pm \sqrt{25 + 16}}{2} = \frac{5 \pm \sqrt{41}}{2}$$
$$= 5.7 \text{ or } -.7, \text{ to one decimal place.}$$

* We do not plot imaginary numbers. A graph is the locus of the points corresponding to all real solutions of an equation. For example, the equation $x^2 + y^2 + 1 = 0$ has no graph.

We find corresponding values of y by substituting these values of x in the equation $y = x - 4$. When $x = 5.7$ we have $y = 5.7 - 4 = 1.7$, and when $x = -.7$ we have $y = -.7 - 4 = -4.7$. These results agree with the values obtained from the graphs.

EXERCISES

Find the real solutions of the following simultaneous equations to one place of decimals by drawing the graphs of the equations and measuring the coördinates of points of intersection. Check by solving algebraically.

solve by alg. then plot

1. $x + 2y = 6,\quad x - y = 3.$

2. $3x + y = 6,\quad x + y = 2.$

3. $2x + 4y + 5 = 0,\quad 3x - y - 3 = 0.$

4. $3x + 2y = 2,\quad x + \frac{1}{3}y = 0.$

5. $2y^2 = x,\quad x - y = 1.$

6. $x^2 = 16y,\quad x + 2y + 2 = 0.$

7. $x^2 + y^2 = 25,\quad 2x + y = 2.$

8. $x^2 + y^2 = 100,\quad x + y = 2.$

9. $x^2 - y^2 = 4,\quad 3x - y = 6.$

10. $xy = 8,\quad x - y + 2 = 0.$

11. $x^2 + 4y^2 = 16,\quad x^2 - y^2 = 9.$

12. $x^2 + y^2 = 9.\quad 9x^2 + 16y^2 = 144.$

13. $y^2 = 16x + 48,\quad 4x^2 + y^2 = 36.$

14. $2xy = 5,\quad x^2 - 4y^2 = 24.$

Show from their graphs that the following simultaneous equations have no real solutions.

15. $x + y = 10,\quad y^2 + x = 0.$

16. $x^2 - 9y^2 = 9,\quad y = 2x.$

★ 6. Oblique coördinates. The two coördinate axes are sometimes taken so that they are not at right angles to each other. Instead of dropping perpendiculars from P to the axes we draw PM parallel to the y-axis and PN parallel to the x-axis, M and N being the points where these lines meet the axes. Then x, the abscissa of P, is the length $\overline{NP}$ if the direction from N to P is the same as that from O to X, otherwise $x = -\overline{NP}$; similarly y, the ordinate of P, is the

length $\overline{MP}$ if the direction from M to P is the same as that from O to Y, otherwise $y = -\overline{MP}$.

Both rectangular and oblique coördinates are included under the term **Cartesian coördinates.**

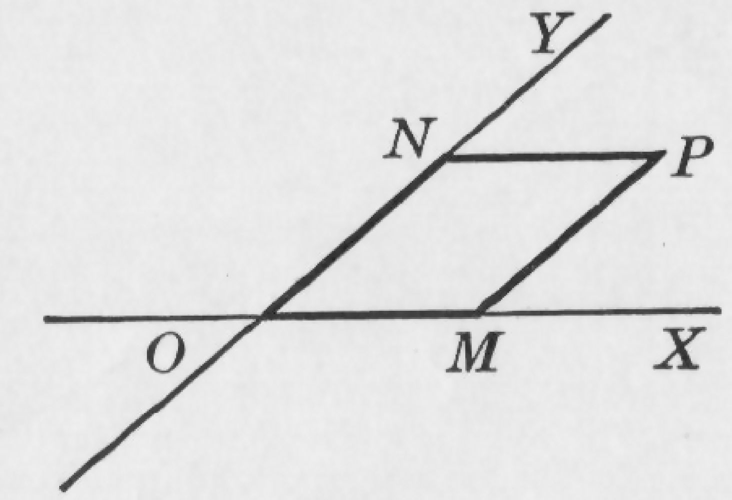


FIG. 11

Although there are problems which are most conveniently studied by the use of oblique coördinates, the only Cartesian coördinates used hereafter in this book, except in a few exercises, will be rectangular.

POLAR COÖRDINATES

7. Polar coördinates defined. Cartesian coördinates locate a point in terms of its distances from two axes. Polar coördinates are referred to a single ray, OA, called the **polar axis** (or **initial line**) with its end-point at the **pole** (or **origin**) O; they locate a point P in terms of its *distance* from O, and its *direction* as seen from O.

In the simplest definition of the polar coördinates * (r, θ) of P, we choose units of measurement and take r as the length of the line OP, and θ as the angle AOP. Here r is positive (except that $r = 0$ for the point O), and is completely determined for a given point. The angle θ may be measured in degrees or in radians. The coördinate r is called the **radius vector** of P, and θ is termed the **vectorial angle** of P.

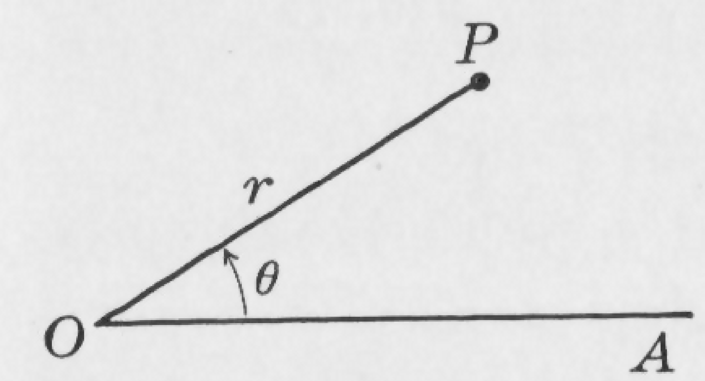


FIG. 12

In another definition of polar coördinates, r is permitted to have negative values. For this definition we prolong OP through O and take θ as an angle from OA either to OP, or

* θ is the Greek letter "theta" (see page 9).

to the prolongation of OP through O. In the former case r is the length $\overline{OP}$, in the latter $r = -\overline{OP}$. Thus in Figure 13 polar coördinates of P are, as there shown, $(-1, -30°)$; other polar coördinates of the same point are $(-1, 330°)$, $(1, 150°)$, $(1, -210°)$. In more general notation, coördinates of a point P given as (r, θ) may also be $(r, 360° + \theta)$, $(-r, 180° + \theta)$, $(-r, -180° + \theta)$.

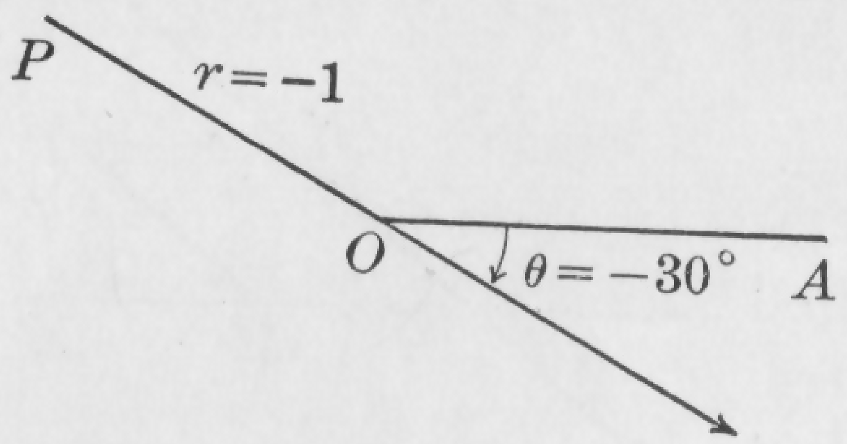


FIG. 13

Hereafter in this book we shall use the definition which permits r to be negative.

8. Relations between rectangular and polar coördinates. If the polar axis OA is taken on the positive x-axis of a rectangular set of axes, as shown in Figure 14, it follows directly from the definitions of the trigonometric functions (page 7) that $\cos\theta = x/r$, $\sin\theta = y/r$. These equations may be written

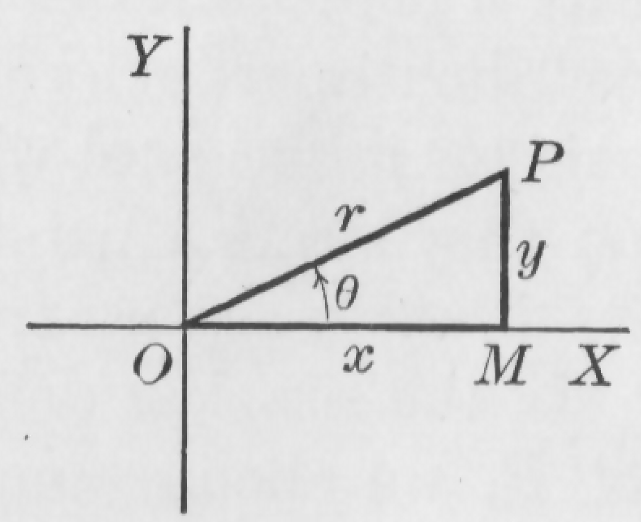


FIG. 14

$$(1)\qquad \begin{aligned} x &= r\cos\theta,\\ y &= r\sin\theta; \end{aligned}$$

they are true no matter in what quadrant P lies.*

To express r and θ in terms of x and y, we derive directly from the right triangle OMP the relations

$$(2)\qquad \begin{aligned} r^2 &= x^2 + y^2, & \tan\theta &= \frac{y}{x},\\ \sin\theta &= \frac{y}{r}, & \cos\theta &= \frac{x}{r}. \end{aligned}$$

* These equations hold even when r is negative; for if $r = -r'$, and $(-r', \theta)$ are coördinates of P, then $(r', 180° + \theta)$ are also coördinates of P, and we have

$$x = r'\cos(180° + \theta) = -r'\cos\theta = r\cos\theta.$$
$$y = r'\sin(180° + \theta) = -r'\sin\theta = r\sin\theta.$$

These results can be written

$$(3)\qquad \begin{aligned} r &= \pm\sqrt{x^2 + y^2},\\ \theta &= \tan^{-1}\frac{y}{x} = \sin^{-1}\frac{y}{r} = \cos^{-1}\frac{x}{r}, \end{aligned}$$

where we use the notation of inverse trigonometric functions; thus $\tan^{-1}(y/x)$ denotes the inverse tangent of y/x, that is, an angle whose tangent is y/x. The second line of (3) states that θ is an angle whose tangent is y/x, whose sine is y/r, and whose cosine is x/r.

Example 1. — Polar coördinates of a point are $(-10, 130°)$. Find the rectangular coördinates.

Solution. — From equations (1) and Table III, of page 11, we have to one decimal place

$$x = -10\cos 130° = -10\cos(180° - 50°) = 10\cos 50° = 6.4,$$
$$y = -10\sin 130° = -10\sin(180° - 50°) = -10\sin 50° = -7.7.$$

Example 2. — Find three different pairs of polar coördinates for the point whose rectangular coördinates are $(-4, 3)$.

Solution. — This point is in the second quadrant, hence θ must terminate in that quadrant if r is positive, and in the fourth quadrant if r is negative. From equations (3) we have

$$r = \pm\sqrt{9 + 16} = \pm 5,$$
$$\theta = \tan^{-1}(-\tfrac{3}{4}) = -\tan^{-1}\tfrac{3}{4} = -37° \pm n\cdot 180°,$$

where n is any integer, or zero. We therefore have the coördinates $(-5, -37°)$, $(5, -217°)$, $(5, 143°)$.

Example 3. — Change from polar to rectangular coördinates in the equation $r = 10\cos\theta$.

Solution. — Multiply both sides by r in order to introduce the combination $r\cos\theta$. We thus have

$$r^2 = 10r\cos\theta,$$

and by means of the first equations of (1) and (2) this reduces to

$$x^2 + y^2 = 10x.$$

Example 4. — Change from rectangular to polar coördinates in the equation $y^2 = 2x + 1$.

Solution. — By means of equations (1) we change this equation to

$$r^2 \sin^2 \theta = 2r \cos \theta + 1,$$

and this reduces to the forms

$$r^2 (1 - \cos^2 \theta) = 2r \cos \theta + 1,$$
$$r^2 = r^2 \cos^2 \theta + 2r \cos \theta + 1,$$
$$r = \pm (r \cos \theta + 1).$$

Hence we have

$$r = \frac{1}{1 - \cos \theta}, \quad \text{or} \quad -r = \frac{1}{1 + \cos \theta}.$$

9. Graphs of equations in polar coördinates. We plot the graph of an equation in polar coördinates as we have done with rectangular coördinates, by locating what seems to be a sufficient number of points and drawing a curve through them. In Chapter VIII we shall improve somewhat on this method and shall discuss curves which are less simple than those of the following Examples and Exercises. We may always *check* a graph by changing the equation in (r, θ) to one in (x, y) by means of equations (1), (2), or (3), and plotting the resulting equation in rectangular coördinates.

For such graphs one may use ordinary paper, laying off angles θ with a protractor and measuring distances with the ruled edge of the protractor. It is easier, however, to use polar coördinate paper as in Figures 15 and 16.

Example 1. — Draw a graph of the equation $r = 10 \cos \theta$.

Solution. — By using Table III, page 11, we compute the following set of values:

r	θ	r	θ
10	0°	− 1.7	100°
9.4	20°	− 3.4	110°
7.7	40°	− 5.	120°
5.	60°	− 7.7	140°
3.4	70°	− 9.4	160°
1.7	80°	− 10.	180°
0.	90°	− 7.7	220°
7.7	− 40°	7.7	320°

The values of $\cos \theta$ for angles that are not acute are calculated by the aid of the reduction formulas, page 8. Thus

$$\cos 100° = \cos (180° - 80°) = -\cos 80°;$$
$$\cos 220° = \cos (180° + 40°) = -\cos 40°;$$
$$\cos 320° = \cos (360° - 40°) = \cos 40°.$$

To draw Figure 15 we need use only values of θ between 0° and 180°. The point $(7.7, -40°)$ is the same as $(-7.7, 140°)$; similarly, the point

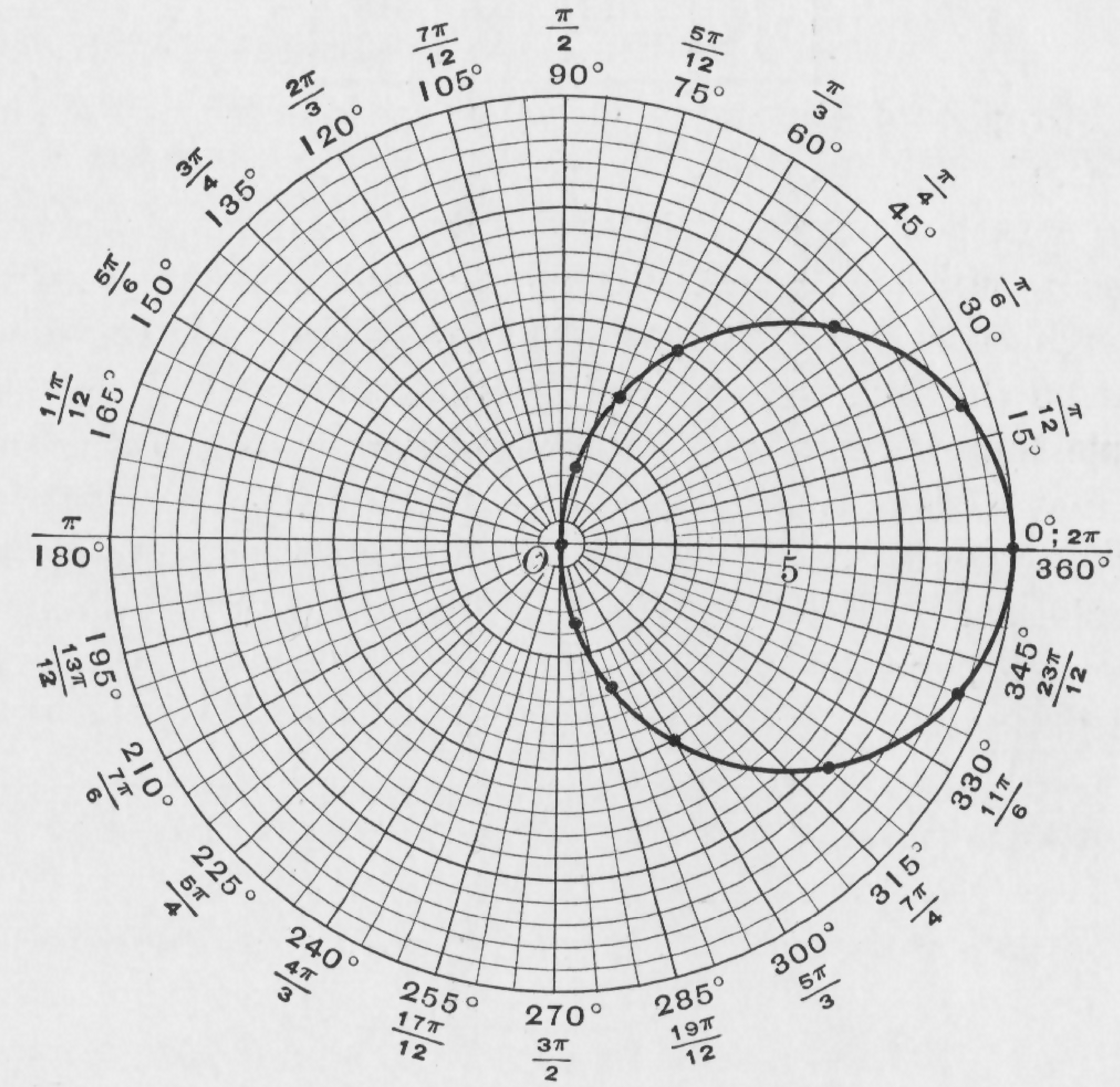


FIG. 15

$(-7.7, 220°)$ is the same as $(7.7, 40°)$; and, in general, if we take θ outside the interval from 0° to 180°, the point (r, θ) thus obtained is one that corresponds to a value of θ in that interval. This last statement is true of many equations in polar coördinates, but by no means of all such equations.

The curve is a circle.

Example 2. — Draw a graph of the equation $r = \dfrac{2}{1 - \cos\theta}$

Solution. — The following is a table of values:

r	θ	r	θ
—	0°	1.	180°
15.	30°	1.1	210°
6.8	45°	1.3	240°
4.	60°	2.	270°
2.1	90°	4.	300°
1.3	120°	6.8	315°
1.1	150°	15.	330°

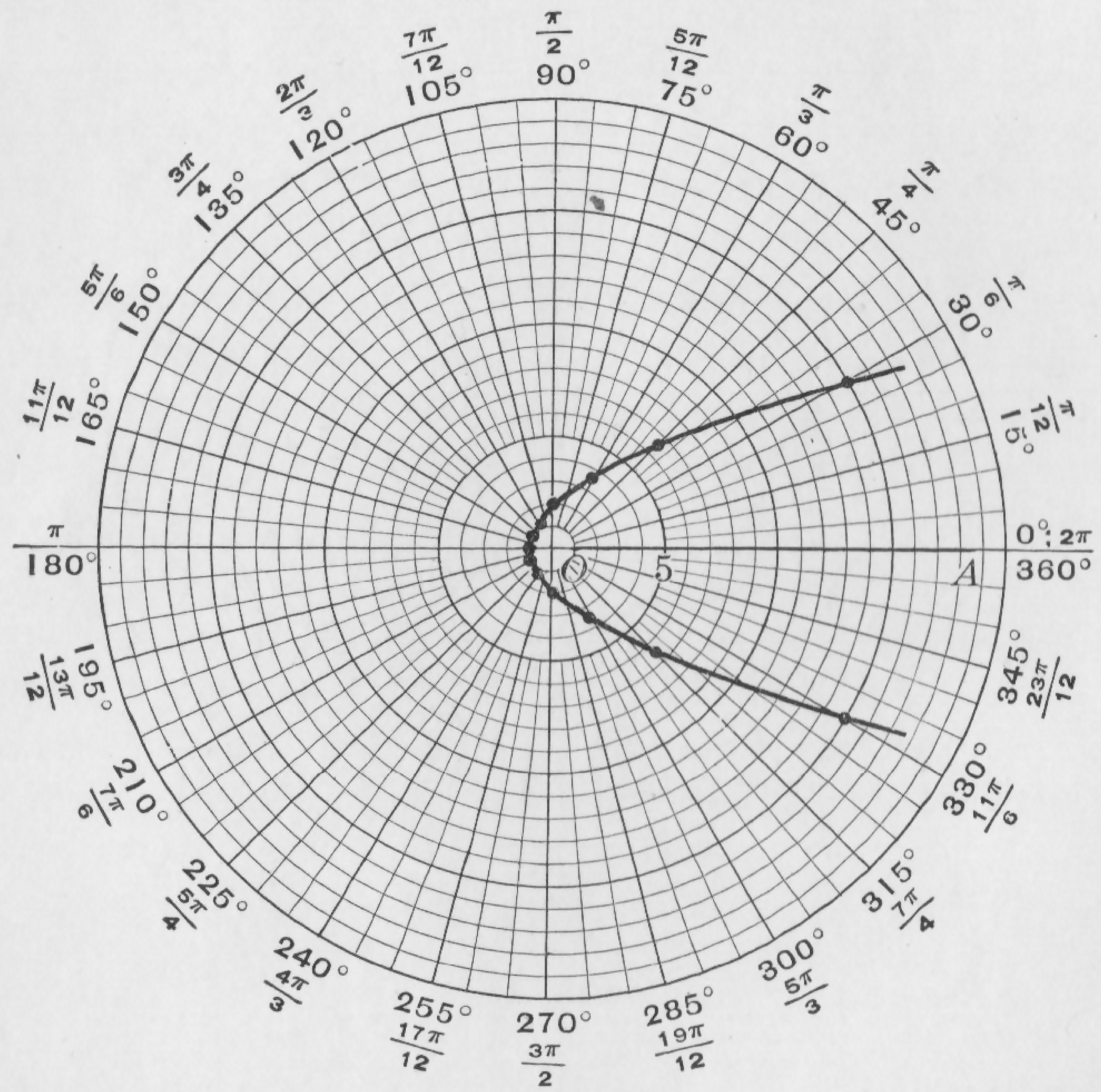


Fig. 16

Since $\cos(\theta \pm n \cdot 360°) = \cos\theta$, it is clear that all points on the curve correspond to values of θ between 0° and 360°.

The curve is a parabola.

EXERCISES

Plot the points whose polar coördinates are as follows (in Exercises 5 and 6, angles are given in radians).

1. $A(6, 45°)$, $B(6, -45°)$, $C(-6, 45°)$, $D(-6, 225°)$.

2. $A(4, 60°)$, $B(4, 300°)$, $C(-4, 60°)$, $D(-4, -60°)$.

3. $A(5, -70°)$, $B(-4, 340°)$, $C(-2, 90°)$, $D(3, 270°)$.

4. $A(-1, -30°)$, $B(4, 225°)$, $C(-2, 180°)$, $D(4, -225°)$.

5. $A(4, \frac{5}{6}\pi)$, $B(-4, \pi)$, $C(6, 0.8)$, $D(-6, 2)$.

6. $A(5, -\frac{3}{2}\pi)$, $B(-5, -\frac{3}{4}\pi)$, $C(3, 0.6)$, $D(-3, 3)$.

For each of the following points give three other sets of polar coördinates, including two for which r *is negative.*

7. $A(4, 45°)$, $B(1, -40°)$, $C(2, 270°)$.

8. $A(10, 30°)$, $B(2, -60°)$, $C(5, 180°)$.

Give the rectangular coördinates of the points in preceding Exercises as indicated.

9. Exercise 1. **10.** Exercise 2. **11.** Exercise 3.

12. Exercise 4. **13.** Exercise 5. **14.** Exercise 6.

Give two sets of polar coördinates in which θ is between 0° and 360° for each of the points whose rectangular coördinates are as follows.

15. $A(2, 2)$, $B(1, -1)$, $C(-3, 4)$, $D(-3, -4)$.

16. $A(4, 4)$, $B(-1, 1)$, $C(5, -10)$, $D(-5, -10)$.

17. $A(0, 3)$, $B(-3, 0)$, $C(1, -2.5)$, $D(-3.2, -4.5)$.

18. $A(2, 0)$, $B(0, -2)$, $C(-.2, .8)$, $D(-2.1, -1.7)$.

Change to polar coördinates in each of the following equations, simplifying the resulting equations where this is possible.

19. $y = x$. **20.** $2y - 3x = 0$.

21. $x = a$. **22.** $y = b$.

23. $ax + by = c$. **24.** $\dfrac{x}{a} + \dfrac{y}{b} = 1$.

25. $x^2 + y^2 - x = 0$.
26. $x^2 + y^2 - y = 0$.
27. $y^2 - 4x = 0$.
28. $x^2 - y = 0$.
29. $x^2 - y^2 = 16$.
30. $x^2 + 2y^2 = 16$.
31. $2xy = 5$.
32. $(x^2 + y^2)^2 = x^2 - y^2$.

Draw graphs of the following equations. After the figure is drawn, change each equation to one in rectangular coördinates, then name the curve.

33. $\theta = 1$.
34. $\theta = -1$.
35. $r = 10$.
36. $r = -10$.
37. $r \cos \theta = 4$.
38. $r \sin \theta = -4$.
39. $r(\cos \theta + 2 \sin \theta) = 4$.
40. $r(\sin \theta - 2 \cos \theta) = -4$.
41. $r = 10 \sin \theta$.
42. $r = \dfrac{2}{1 - \sin \theta}$.
43. $r^2 \sin 2\theta = 25$.
44. $r^2 \cos 2\theta = 16$.
45. $r = 10 (\sin \theta - \cos \theta)$.
46. $r^2 = \dfrac{64}{1 + 3 \sin^2 \theta}$.

CHAPTER II

PRELIMINARY FORMULAS

10. Directed line segments. A line is said to be **directed** if we call all its segments measured in one direction positive and all segments measured in the opposite direction negative. Segments on such a line are also said to be directed.

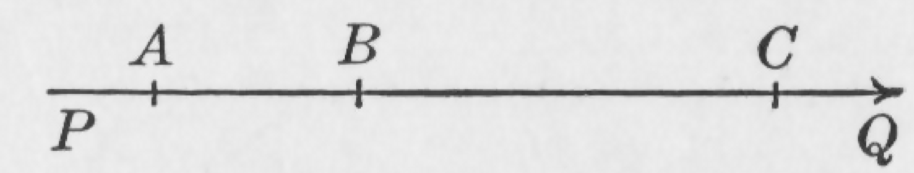


FIG. 17

When we designate a directed segment by AB, we mean that this segment is to be measured in the direction from A to B. Thus in Figure 17 the line PQ has its positive direction indicated by the arrowhead; the segments AB, BC, AC are measured in the positive direction on PQ, and hence are called positive segments, while BA, CB, CA are negative.

When a unit of measure has been chosen, positive segments on a directed line are measured by positive numbers and negative segments by negative numbers. Thus in Figure 17, if the length of segment AB is the unit, the measure of segment AB is 1, of AC is 3, of CB is -2.

We shall hereafter use the symbol AB to denote either the directed segment AB or its measure, but where necessary we shall indicate which is meant. We shall use the symbol $\overline{AB}$ for the length (never negative) of AB, as in Chapter I.

If AB is *positive*, then $AB = \overline{AB}$.
If AB is *negative*, then $AB = -\overline{AB}$,
In either case $AB = -BA$.

11. Directed segments on a coördinate axis. The coördinate axes are directed lines, and coördinates are the measures of directed segments on the axes. Thus in Figure 4, page 13, the coördinates (x, y) of P are the measures of the directed segments OM and ON. In the notation of § 10,

$$x = OM, \qquad y = ON.$$

Let M_1 and M_2 be two points on the x-axis with abscissas x_1 and x_2, so that

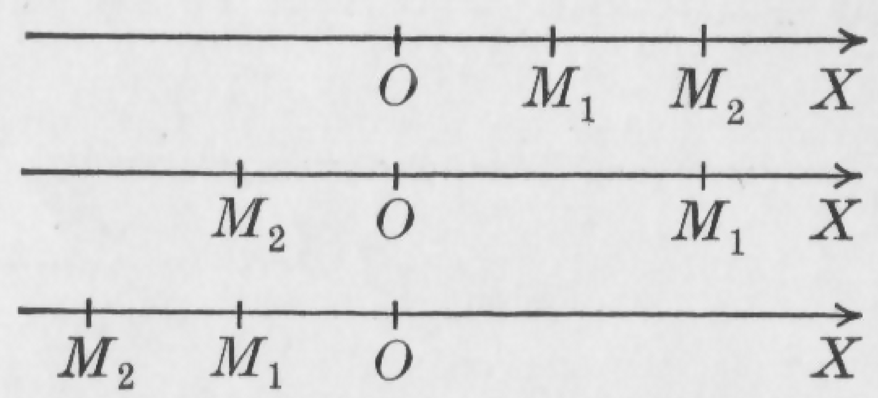


Fig. 18

$$(1) \qquad x_1 = OM_1, \qquad x_2 = OM_2.$$

If the points O, M_1, M_2 have the relative positions shown in the first part of Figure 18, it is clear that

$$(2) \qquad M_1M_2 = OM_2 - OM_1.$$

Moreover, this equation is true, not merely for this case, but for all other relative positions of O, M_1, M_2. For example, if they are as shown in the second figure, then

$$\begin{aligned} M_1M_2 &= M_1O + OM_2 = -OM_1 + OM_2 \\ &= OM_2 - OM_1; \end{aligned}$$

and for the third figure

$$\begin{aligned} M_1M_2 &= -M_2M_1 = -(M_2O - M_1O) = -M_2O + M_1O \\ &= OM_2 - OM_1. \end{aligned}$$

The points O, M_1, M_2 may be arranged in any one of three other orders, and two or all three points may coincide. The student should draw figures and verify relation (2) for each of these cases.

From equations (1) and (2) we have

$$(3) \qquad M_1M_2 = x_2 - x_1.$$

Similarly, if N_1 and N_2 are two points on the y-axis with ordinates y_1 and y_2, then

$$(4) \qquad N_1N_2 = y_2 - y_1.$$

12. Distance between two points. Let $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$ be any two points. Drop perpendiculars to the axes from P_1 and P_2; let the feet of these perpendiculars be M_1, M_2 on the x-axis, and N_1, N_2 on the y-axis. Let Q be the intersection of the lines P_1N_1 and P_2M_2, as shown in Figure 19.

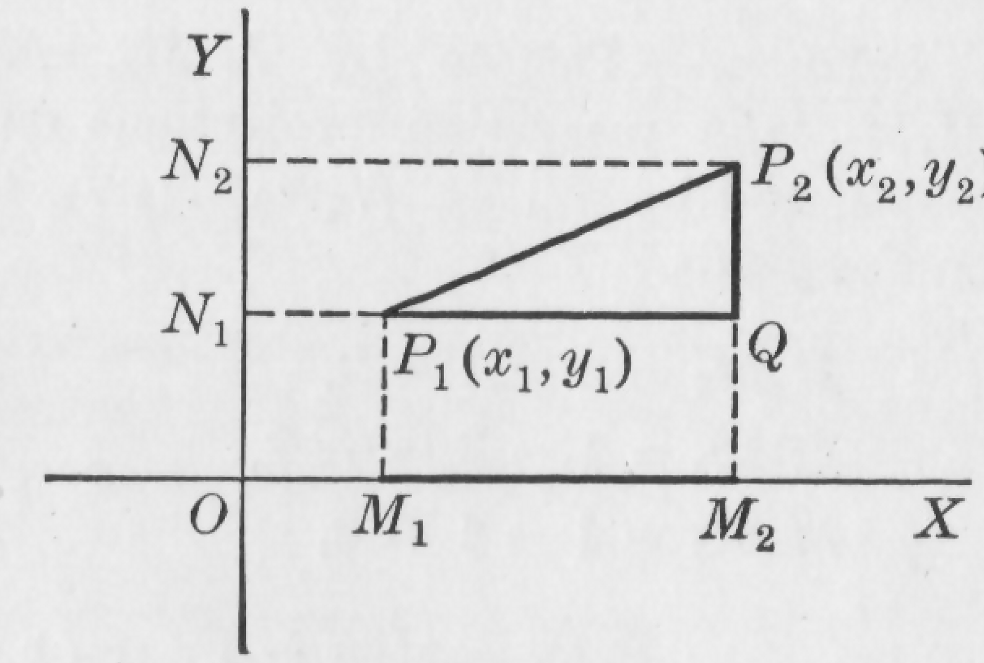


Fig. 19

Applying the theorem of Pythagoras to the right triangle P_1QP_2, we have

$$(1) \qquad \overline{P_1P_2}^2 = \overline{P_1Q}^2 + \overline{QP_2}^2.$$

Obviously $\overline{P_1Q} = \overline{M_1M_2}$, $\overline{QP_2} = \overline{N_1N_2}$. Hence, by formulas (3) and (4) of § 11, we have

$$(2) \qquad \begin{aligned} \overline{P_1Q}^2 &= \overline{M_1M_2}^2 = (x_2 - x_1)^2, \\ \overline{QP_2}^2 &= \overline{N_1N_2}^2 = (y_2 - y_1)^2. \end{aligned}$$

From (1) and (2) it follows that

$$(3) \qquad \overline{P_1P_2} = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}.$$

Since

$$(x_2 - x_1)^2 = (x_1 - x_2)^2, \qquad (y_2 - y_1)^2 = (y_1 - y_2)^2,$$

we also have

$$\overline{P_1P_2} = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}.$$

If P_1 and P_2 are on a line parallel to a coördinate axis, P_1QP_2 is no longer a triangle and our proof of (3) is not valid. One readily verifies, however, that the formula is still correct. It is thus true without exception.

Example. — For the triangle whose vertices are $P_1(1, -1)$, $P_2(3, 1)$, $P_3(1, 3)$ find the measures of the projections* of the directed segments P_1P_2, P_2P_3, P_3P_1 on the coördinate axes; also find the lengths of the sides of the triangle.

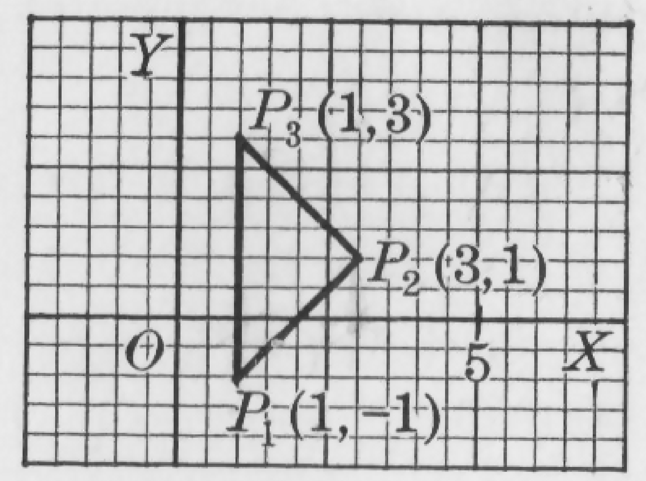


FIG. 20

Solution. — Denote by M_1M_2, M_2M_3, M_3M_1, the respective projections on the x-axis, and by N_1N_2, N_2N_3, N_3N_1, those on the y-axis. Then

$$M_1M_2 = 3 - 1 = 2, \qquad N_1N_2 = 1 - (-1) = 2,$$
$$M_2M_3 = 1 - 3 = -2, \qquad N_2N_3 = 3 - 1 = 2,$$
$$M_3M_1 = 1 - 1 = 0, \qquad N_3N_1 = -1 - 3 = -4,$$

$$\overline{P_1P_2} = \sqrt{[3 - 1]^2 + [1 - (-1)]^2} = 2\sqrt{2},$$
$$\overline{P_2P_3} = \sqrt{[1 - 3]^2 + [3 - 1]^2} = 2\sqrt{2},$$
$$\overline{P_3P_1} = \sqrt{[1 - 1]^2 + [-1 - 3]^2} = 4.$$

The triangle $P_1P_2P_3$ is isosceles, since two of its sides are equal, and it is a right triangle since $\overline{P_1P_2}^2 + \overline{P_2P_3}^2 = \overline{P_3P_1}^2$.

EXERCISES

A figure should be drawn for each exercise.

Find the measures of the projections on the coördinate axes of the directed segments AB, BC, CA *when points* A, B, C *are as follows.*

1. $A(3, -4)$, $B(2, 0)$, $C(-6, -8)$.
2. $A(-5, 0)$, $B(-4, -1)$, $C(2, 3)$.
3. $A(0, 0)$, $B(-1, -4)$, $C(3, 0)$.
4. $A(4, 2)$, $B(0, 0)$, $C(-4, -2)$.

* By definition, the projection of the directed segment P_1P_2 on the x-axis is M_1M_2; its projection on the y-axis is N_1N_2 (Fig. 19).

Find the lengths $\overline{AB}$, $\overline{BC}$, $\overline{CA}$, *when* A, B, C *are given as follows.*

5. As in Exercise 1.
6. As in Exercise 2.
7. As in Exercise 3.
8. As in Exercise 4.

Prove by using the distance formula that the following points are vertices of isosceles triangles.

9. (2, 3), (0, 0), (− 3, 2).
10. (− 2, 0), (0, 4), (5, − 1).
11. (1, 5), (5, 1), (− 3, − 3).
12. (− 4, − 1), (− 2, 3), (1, − 1).

Prove by using the distance formula that the following points are vertices of right triangles.

13. (2, 1), (0, 3), (− 2, 1).
14. (0, 0), (2, 3), (6, − 4).
15. (− 3, − 1), (− 5, 3), (7, 4).
16. (2, − 5), (3, 6), (− 3, 0).

Prove that the following points are vertices of parallelograms; find whether each parallelogram is a rectangle or not. Proofs should not depend upon mere inspection of the figure; use distance relations.

17. (− 1, 5), (− 2, 3), (− 1, 1), (0, 3).
18. (0, 0), (2, − 2), (− 2, − 6), (− 4, − 4).
19. (− 1, − 2), (0, 1), (− 3, 2), (− 4, − 1).
20. (1, − 1), (− 1, 0), (− 4, 6), (− 2, 5).

In the following problems, the coördinates are oblique.

★ **21.** Prove that if ω is the angle from OX to OY, the distance from $P_1(x_1, y_1)$ to $P_2(x_2, y_2)$ is given by the formula

$$\overline{P_1P_2}^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2 + 2(x_2 - x_1)(y_2 - y_1)\cos\omega.$$

★ **22.** If the angle from OX to OY is 60°, find the distance from (3, − 4) to (5, 0).

13. Inclination and slope of a line. The direction of a line l with reference to the axes may be described by giving the angle α which it makes with the x-axis. If the line l is parallel to the x-axis we take $\alpha = 0$; if l has any other direction, we take the angle α as the least positive angle from OX to the line l, that is, the least positive angle through

which OX may be rotated to bring it into coincidence with l. This angle α is called the **inclination** of the line l.

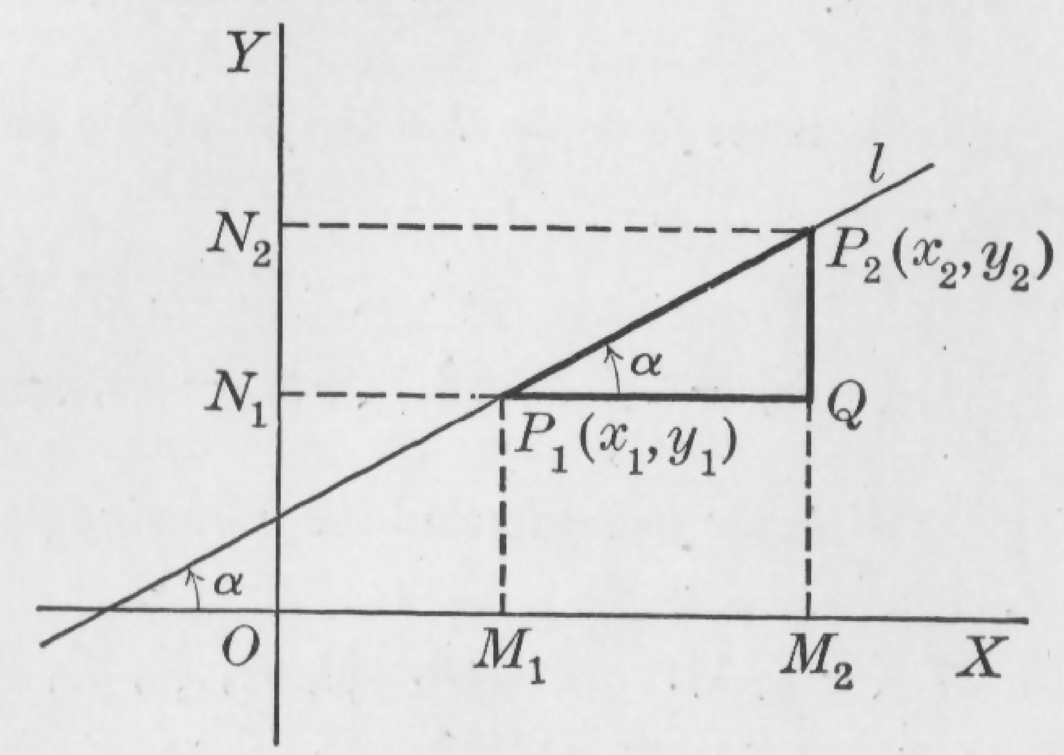


FIG. 21

Another way of describing the direction of a line is to give its **slope,** m, which by definition is the trigonometric tangent of the angle of inclination,

$$(1) \qquad m = \tan \alpha.$$

We now prove the following formula:

The line joining two points $P_1(x_1, y_1)$, $P_2(x_2, y_2)$ *for which* $x_2 \neq x_1$ *has the slope*

$$(2) \qquad m = \frac{y_2 - y_1}{x_2 - x_1}.$$

We have, from the definition (1) and Figure 21,

$$(3) \qquad m = \tan \alpha = \frac{QP_2}{P_1Q} = \frac{N_1N_2}{M_1M_2}.$$

But we have proved (§ 11, equations (4) and (3), p. 33) that

$$N_1N_2 = y_2 - y_1, \qquad M_1M_2 = x_2 - x_1.$$

Substituting these expressions in (3), we obtain (2).

For Figure 22 we have

$$m = \tan \alpha = \frac{RP_1}{P_2R} = \frac{-QP_2}{-P_1Q},$$
$$= \frac{QP_2}{P_1Q} = \frac{N_1N_2}{M_1M_2}.$$

Hence equation (2) follows as before.

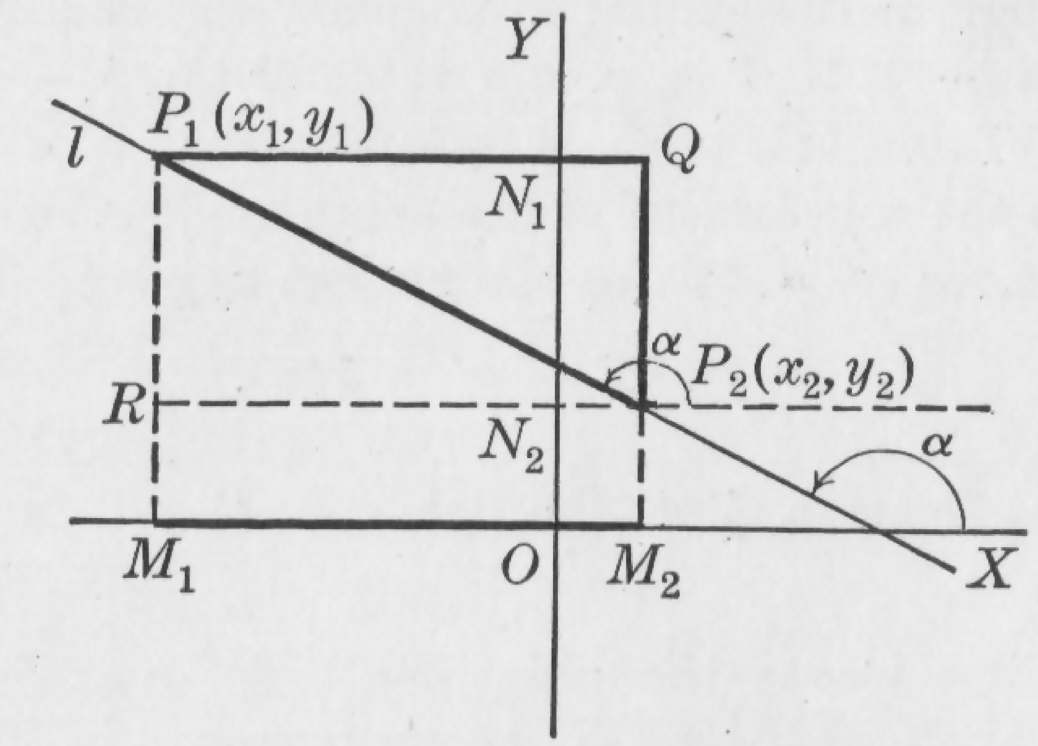


FIG. 22

If we change signs in numerator and denominator of the fraction in (2) we obtain

$$(4) \qquad m = \frac{y_1 - y_2}{x_1 - x_2}.$$

Since (4) is equivalent to (2) we may use either to obtain the slope of the line passing through P_1 and P_2.

If $x_2 = x_1$, the line P_1P_2 is parallel to the y-axis and its inclination is 90°; such a line has no slope.* A line whose inclination is nearly 90° has a slope which is numerically large.

If the slope m is positive, the inclination α is acute; if m is negative, α is obtuse; if $m = 0$, then $\alpha = 0$ and the line is parallel to the x-axis, or coincident with it.

* Such a line is sometimes said to have an *infinite* slope, $m = \infty$. This last equation is, however, merely symbolic.

Example 1. — Find the slope and inclination of the line through (1, 0) (4, − 2).

Solution. — By formula (4),

$$m = \frac{0 - (-2)}{1 - 4} = -\frac{2}{3} = \tan \alpha.$$

Hence the slope is $-\frac{2}{3}$ and the inclination α is an angle whose tangent is $-\frac{2}{3}$. This angle is designated in trigonometry as a value of the inverse tangent of $-\frac{2}{3}$, that is, α is a value of $\tan^{-1}(-\frac{2}{3})$. To find α with the aid of Table III, page 11, we note that α is in the second quadrant and is the supplement of the acute angle α' whose tangent is $\frac{2}{3} = .667$. We have $\alpha' = 34°$ (to the nearest degree). Hence

$$\alpha = 180° - \alpha' = 146°.$$

Example 2. — Draw a line through (2, 4) with slope $-\frac{3}{4}$.

Solution. — First locate the point $P_1(2, 4)$. Next plot a point P_2 such that (in the notation of Figures 21 and 22)

$$\frac{QP_2}{P_1Q} = -\frac{3}{4}.$$

To do this draw $P_1Q = 4$, then $QP_2 = -3$, as in Figure 23. The line P_1P_2 is the required line.

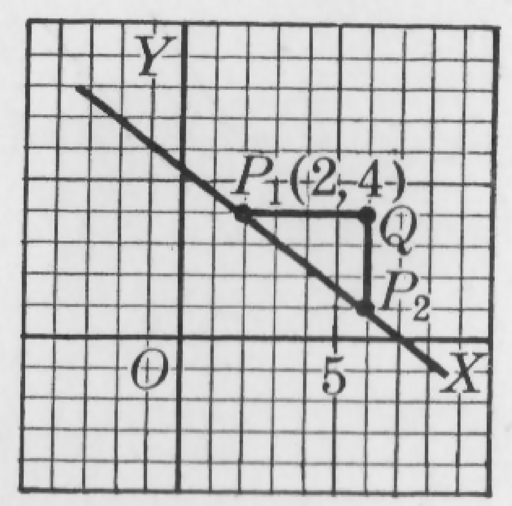


FIG. 23

EXERCISES

Prove the formula for the slope of the line joining $P_1(x_1, y_1)$ *and* $P_2(x_2, y_2)$, *with figures as follows.*

1. P_1 in the first quadrant, P_2 in the second, y_2 greater than y_1.
2. P_1 in the second quadrant, P_2 in the fourth.
3. P_1 in the fourth quadrant, P_2 in the second.
4. P_1 in the third quadrant, P_2 in the fourth, y_2 less than y_1.

Find the slope and the inclination of each of the lines through the following points.

5. (*a*) (− 2, − 2), (3, 3); (*b*) (− 1, 0), (0, − 1); (*c*) (2, − 5) (4, 1); (*d*) (− 1.6, − 2.4), (3, 5).

6. (*a*) (2, − 3), (6, 1); (*b*) (2, 3), (− 1, 3); (*c*) (− 3, 4), (5, 0); (*d*) (− .6, 2.8), (− 2, 1).

Draw lines through the following points, with the slopes indicated.

7. (0, 2); $m = 1$.
8. (− 1, 0); $m = 4$.
9. (− 3, 2); $m = \frac{1}{4}$.
10. (− 2, − 4); $m = -\frac{3}{2}$.
11. (2, 4); $m = -\frac{1}{10}$.
12. (6, − 2); $m = -10$.

14. Parallel lines. Perpendicular lines. When two lines l_1 and l_2 are parallel they have the same inclination and therefore the same slope.* If the two lines l_1 and l_2 are mutually perpendicular, one of them must have an inclination 90° greater than that of the other. If α_1 is the inclination of l_1 and α_2 that of l_2, we have either

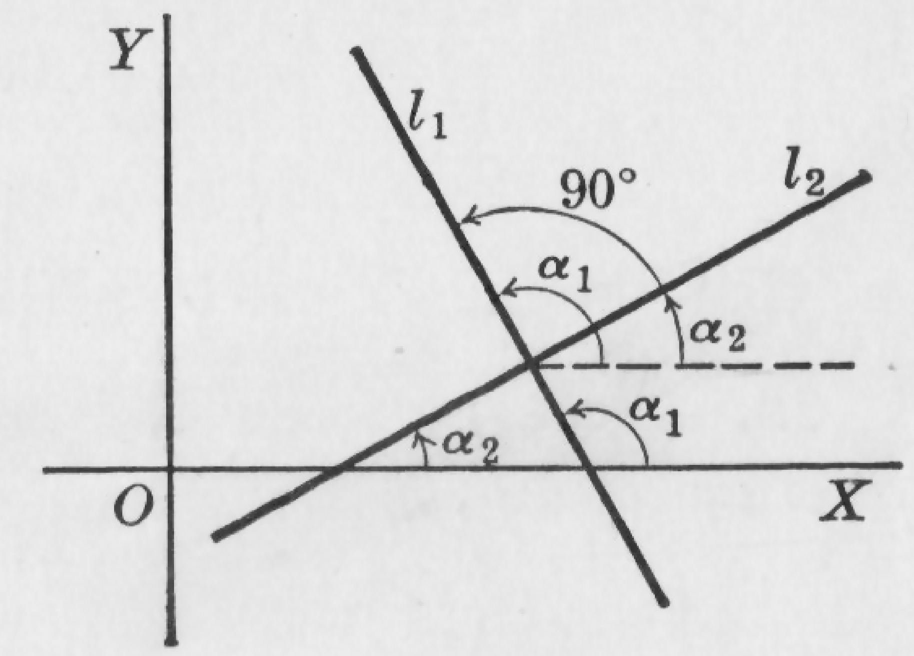


FIG. 24

$$(1) \qquad \alpha_1 = 90° + \alpha_2$$

or

$$(2) \qquad \alpha_2 = 90° + \alpha_1.$$

If (1) holds, then

$$m_1 = \tan \alpha_1 = \tan (90° + \alpha_2)$$
$$= -\cot \alpha_2 = -\frac{1}{\tan \alpha_2} = -\frac{1}{m_2};$$

if (2) holds, we have

$$m_2 = \tan \alpha_2 = -\cot \alpha_1 = -\frac{1}{m_1}.$$

In either case, the product of the slopes, m_1m_2, is − 1.

* An exception to this statement occurs when both lines are parallel to the y-axis, and therefore have no slope. Similar exceptions should be made later in this section and elsewhere when formulas are given involving slopes.

We summarize these results as follows:

If two lines of slopes $\mathbf{m_1}$, $\mathbf{m_2}$ *are parallel, their slopes are equal;* $\mathbf{m_1 = m_2}$.

If two lines of slopes $\mathbf{m_1}$, $\mathbf{m_2}$ *are perpendicular to each other, the product of their slopes is equal to* -1; $\mathbf{m_1 m_2 = -1}$.

The converse of each of these statements is also true.

To prove this last statement we observe that if $m_1 = m_2$, then the lines have the same inclination and hence are parallel. If $m_1 m_2 = -1$, the argument above, on perpendicular lines is reversible, with the conclusion that either

$$\alpha_1 = 90° + \alpha_2$$

or

$$\alpha_2 = 90° + \alpha_1,$$

so that the lines must be perpendicular.

15. Angle between two intersecting lines. Two intersecting lines l_1, l_2, of inclinations α_1, α_2, make with each other

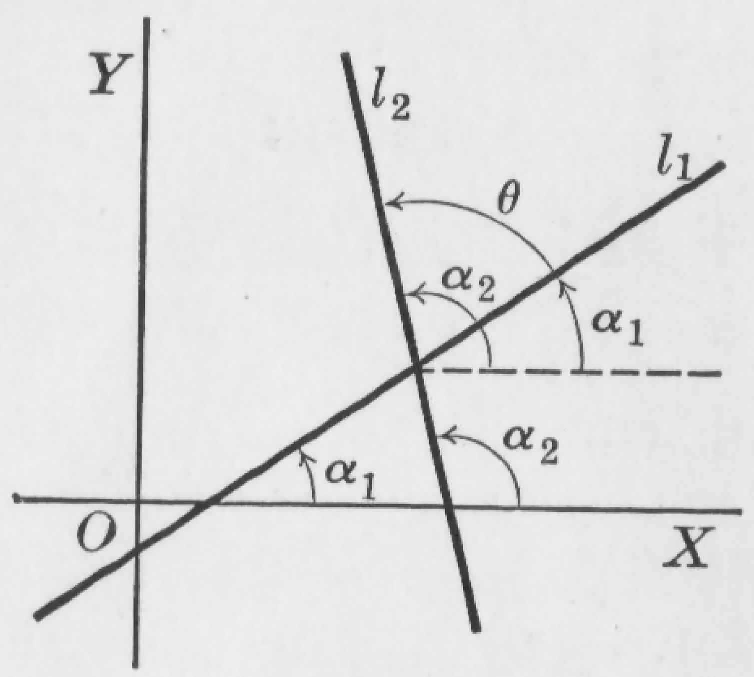


FIG. 25

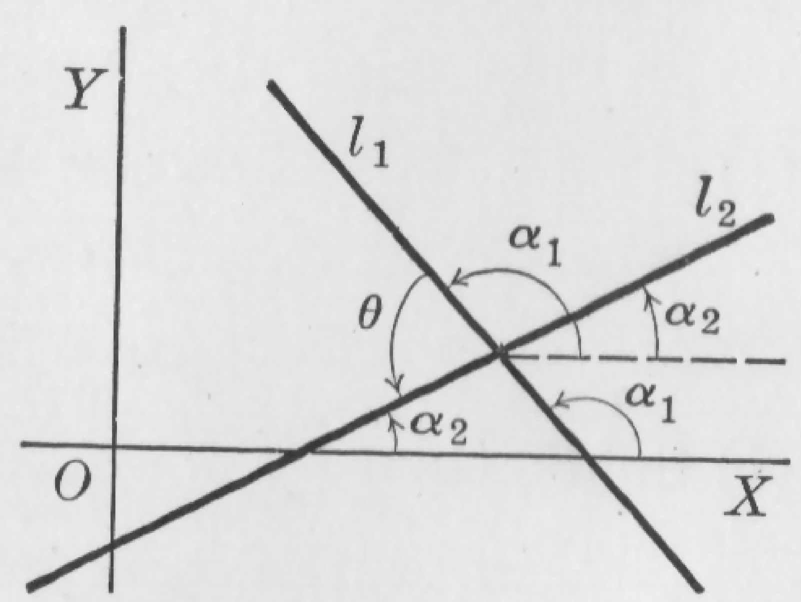


FIG. 26

various positive and negative angles. In order to be specific we define *the angle* θ *from* l_1 *to* l_2 as the positive angle, less than 180°, through which l_1 must be rotated in order to coincide with l_2.

The following formula holds, both when α_2 is greater than α_1, as in Figure 25, and when α_2 is less than α_1, as in Figure 26:

$$(1) \qquad \tan\theta = \frac{m_2 - m_1}{1 + m_1 m_2}.$$

The proof of this formula for the case of Figure 25, where $\theta = \alpha_2 - \alpha_1$, depends on the use of the last addition formula of page 8. We have

$$\tan\theta = \tan(\alpha_2 - \alpha_1) = \frac{\tan\alpha_2 - \tan\alpha_1}{1 + \tan\alpha_1 \tan\alpha_2} = \frac{m_2 - m_1}{1 + m_1 m_2}.$$

In the case of Figure 26 we have

$$\alpha_1 - \alpha_2 + \theta = 180°,$$
$$\theta = 180° + (\alpha_2 - \alpha_1).$$

Since $\tan(180° + A) = \tan A$ (see a reduction formula, page 8), it follows that

$$\tan\theta = \tan(\alpha_2 - \alpha_1),$$

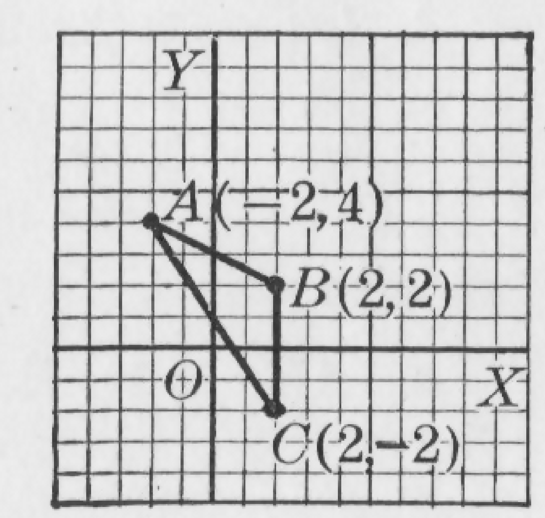
and we complete the proof as above.

An exception presents itself when $\theta = 90°$, since $\tan\theta$ has then no value; this case, where the lines are perpendicular to each other, has been discussed in § 14.

Cases where one of the lines is parallel to the y-axis also present exceptions. If, for example, $\alpha_2 = 90°$, then m_2 does not exist and (1) cannot hold; but here we have the relations $\theta = 90° - \alpha_1$, or $\theta = 270° - \alpha_1$, so that

$$\tan\theta = \cot\alpha_1 = \frac{1}{m_1}.$$

Example 1. — Find the interior angles of the triangle whose vertices are $A(-2, 4)$, $B(2, 2)$, $C(2, -2)$.



Solution. — The slope of the line AB is $\frac{2-4}{2-(-2)} = -\frac{1}{2}$; similarly the slope of AC is $-\frac{3}{2}$. The angle CAB is the angle from AC to AB, hence we have, from formula (1),

$$\tan A = \frac{\text{slope } AB - \text{slope } AC}{1 + (\text{slope } AB)\cdot(\text{slope } AC)} = \frac{-\frac{1}{2} - (-\frac{3}{2})}{1 + (-\frac{1}{2})(-\frac{3}{2})} = \frac{4}{7}.$$

From Table III, page 11, we have

$$A = 30° \text{ (to the nearest degree).}$$

To find angles B and C, note that BC is parallel to the y-axis, and use the relations

$$B = 270° - \text{inclination } AB, \quad C = \text{inclination } AC - 90°,$$

$$\tan B = \frac{1}{\text{slope } AB} = -2, \quad \tan C = \frac{-1}{\text{slope } AC} = \frac{2}{3}.$$

Hence

$$B = 117°, C = 34° \text{ (to the nearest degree in each case).}$$

Example 2. — If l_1 has the slope $\frac{1}{2}$, find the slope of the line l_2 such that the angle from l_1 to l_2 is $45°$.

Solution. — If we designate the slope of l_2 by m_2, we have

$$\tan 45° = \frac{m_2 - \frac{1}{2}}{1 + \frac{1}{2}m_2}.$$

But $\tan 45° = 1$, hence

$$1 = \frac{m_2 - \frac{1}{2}}{1 + \frac{1}{2}m_2},$$

$$1 + \tfrac{1}{2}m_2 = m_2 - \tfrac{1}{2},$$

$$m_2 = 3.$$

EXERCISES

Find the angle from l_1 to l_2 when the following data are given.

1. l_1 has the slope $-\frac{1}{2}$, l_2 has the slope 3.

2. l_1 has the slope 3, l_2 has the slope 0.

3. l_1 passes through points $(-3, 0)$, $(3, 3)$, l_2 passes through $(0, 0)$, $(3, -4)$.

4. l_1 passes through points $(a + b, b)$, $(a, b - a)$, l_2 passes through (a, b), $(0, 0)$.

Find the interior angles of each of the triangles whose vertices are the following points.

5. $A(-1, 1)$, $B(3, 1)$, $C(3, -3)$.

6. $A(-2, 2)$, $B(-2, -2)$, $C(2, -2)$.

7. $A(-2, 1)$, $B(2, -1)$, $C(3, 4)$.

8. $A(-2, -1)$, $B(2, 1)$, $C(5, 0)$.

9. $A(0, 0)$, $B(-1, 2)$, $C(3, 1)$.

10. $A(-1, 0)$, $B(0, -1)$, $C\left(\dfrac{\sqrt{3} - 1}{2}, \dfrac{\sqrt{3} - 1}{2}\right)$.

In each of the following problems find m, the slope of line l, when m_1, the slope of l_1, and θ, the angle from l_1 to l, are as follows.

11. $m_1 = 2$, $\theta = 45°$.

12. $m_1 = -2$, $\theta = 135°$.

13. $m_1 = 1$, $\theta = 90°$.

14. $m_1 = 0$, $\theta = 150°$.

Prove by means of slopes that the following points are vertices of parallelograms; find whether each figure is a rectangle by using the test of § 14.

15. $(-2, -2)$, $(-1, 1)$, $(2, 4)$, $(1, 1)$.

16. $(0, 0)$, $(1, 2)$, $(2, -1)$, $(3, 1)$.

17. $(2, 0)$, $(6, 4)$, $(2, 8)$, $(-2, 4)$.

18. $(-6, -4)$, $(0, -2)$, $(6, 4)$, $(0, 2)$.

19. (a, b), $(a + c, b + d)$, $(a + d, b - c)$, $(a + c + d, b - c + d)$.

20. $(-a, -b)$, (a, b), (c, d), $(c - 2a, d - 2b)$.

16. Point of division. Mid-point. If $P_0(x_0, y_0)$, $P_1(x_1, y_1)$, $P_2(x_2, y_2)$ are three points on a directed line, the point P_0 is said to **divide the segment P_1P_2 in the ratio r_1/r_2**

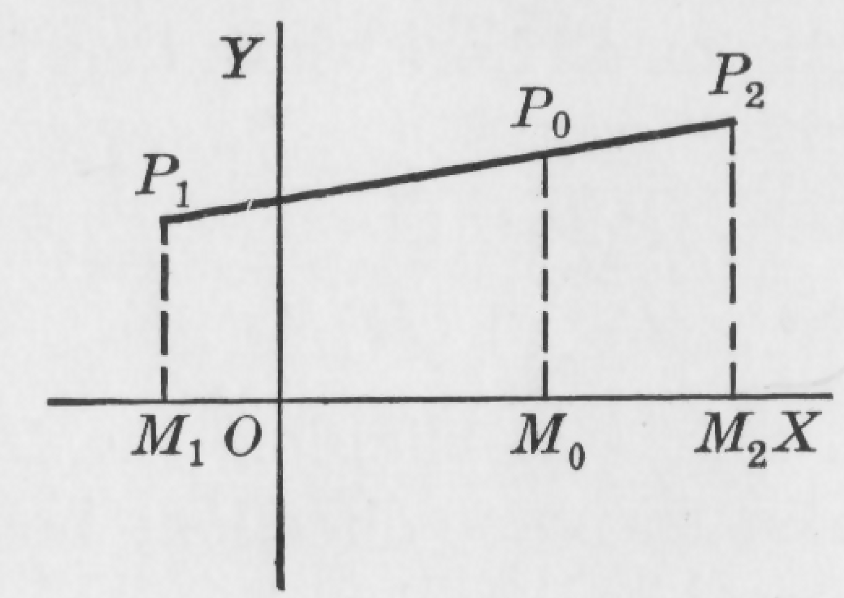


FIG. 27

provided that

$$\frac{P_1P_0}{P_0P_2} = \frac{r_1}{r_2}. \tag{1}$$

We shall prove the following theorem which expresses the coördinates of P_0, the point of division, in terms of r_1, r_2, and the coördinates of P_1 and P_2.

If $P_0(x_0, y_0)$ *divides the directed segment from* $P_1(x_1, y_1)$ *to* $P_2(x_2, y_2)$ *in the ratio* r_1/r_2, *then*

$$x_0 = \frac{r_1x_2 + r_2x_1}{r_1 + r_2}, \quad y_0 = \frac{r_1y_2 + r_2y_1}{r_1 + r_2}. \tag{2}$$

There are two cases to consider; the first is shown in Figure 27, where P_0 is between P_1 and P_2 and is therefore called a point of *internal division*; the second is illustrated in Figure 28, where P_0 is a point of *external division*, lying outside the segment P_1P_2. In both cases we drop perpen-

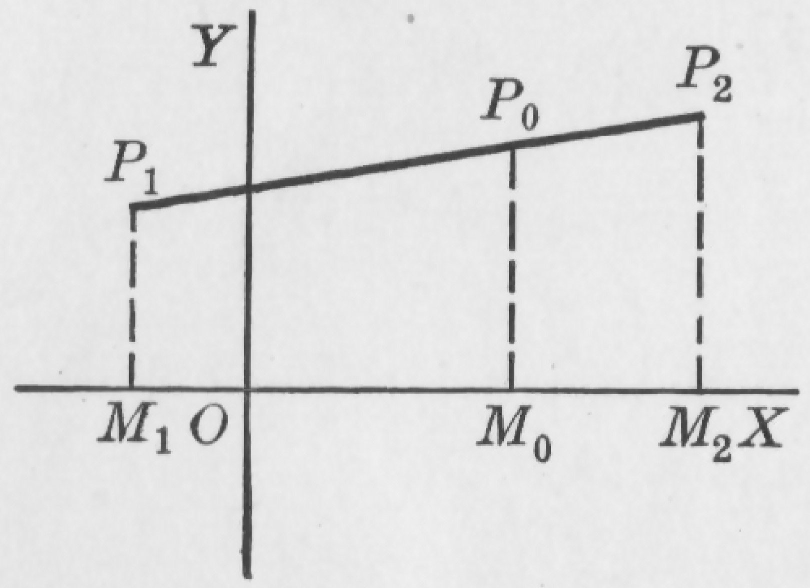


FIG. 27

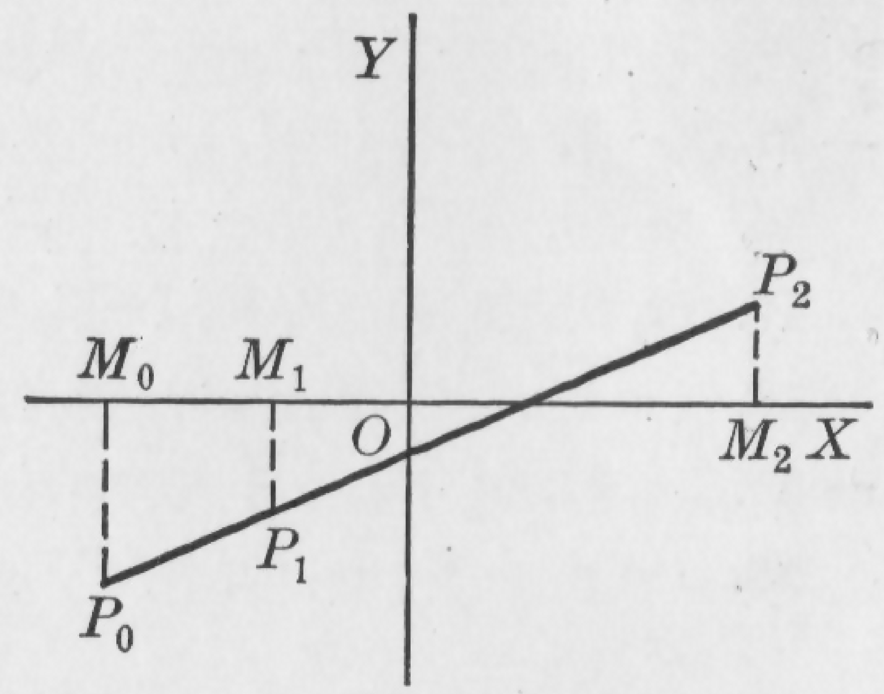


FIG. 28

diculars P_1M_1, P_0M_0, P_2M_2 to the x-axis. These three parallels, according to a theorem of geometry, intercept proportional lengths on all lines which they intersect; hence we have *

$$(3) \qquad \frac{\overline{P_1P_0}}{\overline{P_0P_2}} = \frac{\overline{M_1M_0}}{\overline{M_0M_2}}.$$

If P_0 is a point of internal division (Fig. 27) the segments P_1P_0 and P_0P_2 have the same direction; hence for the ratio of their measures we have

$$(4) \qquad \frac{P_1P_0}{P_0P_2} = \frac{\overline{P_1P_0}}{\overline{P_0P_2}}.$$

Similarly

$$(5) \qquad \frac{M_1M_0}{M_0M_2} = \frac{\overline{M_1M_0}}{\overline{M_0M_2}}.$$

* We recall that $\overline{P_1P_0}$ is the length of the segment from P_1 to P_0, and is positive or zero, while P_1P_0 is the directed segment from P_1 to P_0, or the measure of that segment. In the latter sense, P_1P_0 is equal either to $\overline{P_1P_0}$ or to $-\overline{P_1P_0}$.

From (3), (4), and (5) it follows that

$$(6) \qquad \frac{P_1P_0}{P_0P_2} = \frac{M_1M_0}{M_0M_2}.$$

Equation (6) also holds if P_0 is a point of external division (Fig. 28). Here, however, P_1P_0 has the direction opposite to that of P_0P_2; it turns out that (4) and (5) are replaced by

$$(4') \qquad \frac{P_1P_0}{P_0P_2} = -\frac{\overline{P_1P_0}}{\overline{P_0P_2}}; \qquad \frac{M_1M_0}{M_0M_2} = -\frac{\overline{M_1M_0}}{\overline{M_0M_2}},$$

and (3), with (4′), again gives (6).

Since

$$M_1M_0 = x_0 - x_1, \quad \text{and} \quad M_0M_2 = x_2 - x_0,$$

equation (1) can be written, with the aid of (6),

$$\frac{x_0 - x_1}{x_2 - x_0} = \frac{r_1}{r_2}.$$

We solve for x_0, as follows:

$$\begin{aligned} r_2x_0 - r_2x_1 &= r_1x_2 - r_1x_0, \\ (r_1 + r_2)x_0 &= r_1x_2 + r_2x_1, \\ x_0 &= \frac{r_1x_2 + r_2x_1}{r_1 + r_2}. \end{aligned}$$

We prove the formula for y_0 similarly, dropping perpendiculars to the y-axis instead of the x-axis.

It is to be noted that, from equations (4) and (4′), if P_0 is a point of internal division the ratio r_1/r_2 is positive, and if P_0 is a point of external division r_1/r_2 is negative. The converse is also true.

If P_0 is the mid-point of the segment P_1P_2, then we have $P_1P_0 = P_0P_2$ and $r_1 = r_2$; hence *the coördinates* (x_0, y_0) *of the* **mid-point** *of the segment whose end-points are* (x_1, y_1), (x_2, y_2) *are given by the formulas*

$$(7) \qquad \boldsymbol{x_0 = \tfrac{1}{2}(x_1 + x_2), \qquad y_0 = \tfrac{1}{2}(y_1 + y_2).}$$

Example. — On the line joining $P_1(2, -1)$ and $P_2(-4, 1)$ find the points which are one-half as far from P_1 as P_2 is from P_1.

Solution. — There are two such points. One is the mid-point of the segment P_1P_2, hence its coördinates are given by the equations

$$x_0 = \tfrac{1}{2}(2 - 4) = -1, \qquad y_0 = \tfrac{1}{2}(-1 + 1) = 0.$$

The other point to be found is external to P_1P_2; if it is designated $P_0'(x_0', y_0')$ it is easily seen that segment P_1P_0' is opposite in direction to $P_0'P_2$ and its length is one-third the length of $P_0'P_2$. Hence we have, $r_1/r_2 = P_1P_0'/P_0'P_2 = -1/3$, and we can take $r_1 = -1$, $r_2 = 3$. Formulas (2) give

$$x_0' = \frac{(-1)(-4) + (3)(2)}{-1 + 3} = 5,$$

$$y_0' = \frac{(-1)(1) + (3)(-1)}{-1 + 3} = -2.$$

The two points are $P_0(-1, 0)$ and $P_0'(5, -2)$.

EXERCISES

1. Find the coördinates of each of the points dividing the segment from $(0, -1)$ to $(-2, 3)$ in the following ratios: (*a*) $1/2$; (*b*) $2/1$; (*c*) $1/-2$; (*d*) $-1/2$.

2. Find the coördinates of each of the points dividing the segment from $(-2, 0)$ to $(3, -2)$ in the following ratios: (*a*) $2/3$; (*b*) $3/2$; (*c*) $-2/3$; (*d*) $3/-2$.

3. Prove that if the three points $P_0(x_0, y_0)$, $P_1(x_1, y_1)$, $P_2(x_2, y_2)$ are on a straight line, and if $P_1P_0/P_1P_2 = h/k$, then

$$x_0 = x_1 + \frac{h}{k}(x_2 - x_1), \quad y_0 = y_1 + \frac{h}{k}(y_2 - y_1).$$

Consider the case where P_0 is external to P_1P_2 as well as the case where it is internal.

4. Find the coördinates of a point P_0 which is on the line joining $(-2, -3)$ and $(1, 0)$, if P_0 is three-fourths of the way from $(-2, -3)$ to $(1, 0)$.

5. Find the coördinates of two points P_0, P_0', on the line joining $P_1(-3, 2)$ and $P_2(3, -4)$, if P_0 and P_0' are each three times as far from P_2 as from P_1.

6. Find the coördinates of two points P_0, P_0', on the line joining $P_1(-2, 0)$, $P_2(4, 6)$, if P_0 and P_0' are each four times as far from P_1 as from P_2.

7. Find the coördinates of the points that trisect the segment whose end-points are $(2, -1)$, $(-4, 8)$.

8. Find the mid-points of the sides of the triangle whose vertices are $(-1, 2)$, $(3, 4)$, $(2, -1)$.

9. From a theorem of plane geometry, the medians of a triangle meet in a point two-thirds of the way from a vertex to the mid-point of the side opposite that vertex. Use this theorem to find the coördinates of the point of intersection of the medians of the triangle whose vertices are $(0, 0)$, $(2, 0)$, $(3, -4)$.

10. By the method indicated in the preceding Exercise, find the coördinates of the point of intersection of the medians of the triangle whose vertices are $(-2, 0)$, $(0, 2)$, $(0, -1)$.

11. If the vertices of a triangle are (x_1, y_1), (x_2, y_2), (x_3, y_3), find the coördinates of the point of intersection of the medians, using the theorem of plane geometry stated in Exercise 9.

12. Prove by using the distance formula of page 33 that (x_0, y_0) as given by formula (7), page 45, is the mid-point of the line joining (x_1, y_1), (x_2, y_2).

13. Prove by using the distance formula of page 33 that if (x_0, y_0) is given by formula (2), page 44, then $\overline{P_1P_0}/\overline{P_0P_2} = \pm r_1/r_2$. When are we to take the $+$ sign and when the $-$ sign in this equation?

★ 17. Analytic proof of geometric theorems.

The methods of analytic geometry can be used in the proof of theorems concerning general properties of figures. Often the analytic proof is much briefer than a proof in the style of elementary geometry.

Where a property to be proved is independent of the position of the figure we place our coördinate axes so as to make coördinate relations as simple as possible; for example, the x-axis may be taken along a side of a figure and the origin either at a vertex or at the mid-point of a side. In proving a general proposition true for all figures of a given class we give literal values such as (a, b), (c, d), $\cdots$ to the coördinates of the points that determine the figure, and prove that the proposition holds for all values of $a, b, c, d, \cdots$. The following example will illustrate the procedure.

Example. — Prove that the diagonals of a parallelogram bisect each other.

Solution. — Let the parallelogram be $ABCD$, the vertices being in the order shown in Figure 29. We place the axes so that the origin is at the vertex A, and the side AB is on the x-axis. The coördinates of A are $(0, 0)$; those of B can be written $(a, 0)$. The position of C is unrestricted if we designate C as the point (b, c). The coördinates of D, however, are now determined in terms of a, b, c. Since D and C are equidistant from AB and on the same side of AB, the ordinate of D is c, and since $CD = AB$, the abscissa of D must be $a + b$. The coördinates of the vertices are shown in Figure 29.

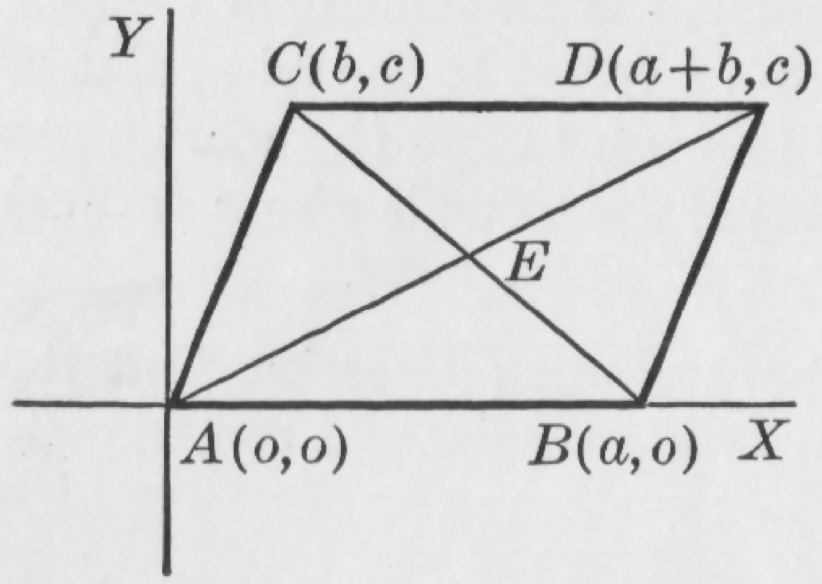


FIG. 29

By formula (7), page 45, the mid-point of AD has the coördinates

$$x = \frac{0 + a + b}{2}, \qquad y = \frac{0 + c}{2};$$

the mid-point of BC has the coördinates

$$x = \frac{a + b}{2}, \qquad y = \frac{0 + c}{2}.$$

Since these mid-points coincide, the diagonals AD and CB bisect each other.

EXERCISES

Prove that if the axes are suitably placed the following statements are true.

1. The vertices of any right triangle may be taken as $(0, 0)$, $(a, 0)$, $(0, b)$.

2. The vertices of any triangle may be taken as $(0, 0)$, $(a, 0)$, (b, c).

3. The vertices of any triangle may be taken as $(a, 0)$, $(b, 0)$, $(0, c)$.

4. The vertices of any rectangle may be taken as $(-a, -b)$, $(a, -b)$, $(-a, b)$, (a, b).

Prove the following propositions analytically.

5. The line segment joining the mid-points of two sides of a triangle is parallel to the third side and has half the length of the third side.

6. The diagonals of a rectangle are equal.

7. The three vertices of a right triangle are equidistant from the mid-point of the hypotenuse.

8. The diagonals of a square are perpendicular to each other.

9. The lines joining mid-points of consecutive sides of a quadrilateral form a parallelogram.

10. If the diagonals of a convex quadrilateral bisect each other, the quadrilateral is a parallelogram.

11. The sum of the squares of the four sides of a parallelogram is equal to the sum of the squares of the diagonals.

POLAR COÖRDINATES

★ **18. Distance between two points.** If polar coördinates of two points P_1, P_2, are (r_1, θ_1), (r_2, θ_2) respectively, the following formula holds for the distance between the points:

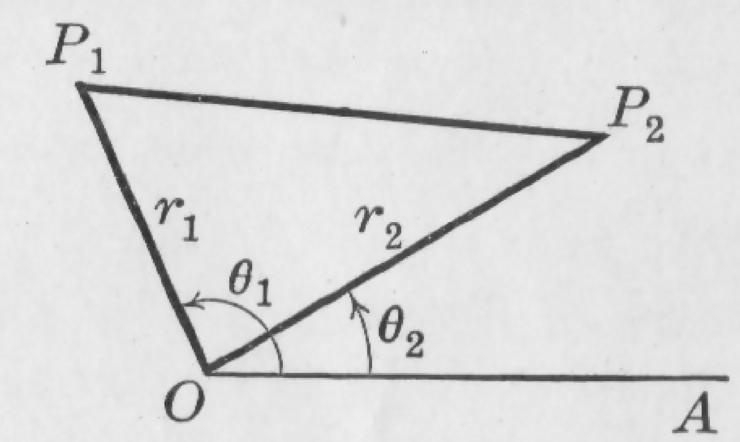


FIG. 30

$$(1) \qquad \overline{P_1P_2}^2 = r_1^2 + r_2^2 - 2r_1r_2 \cos(\theta_1 - \theta_2).$$

This follows from the law of cosines (page 9) applied to the triangle OP_1P_2 (Fig. 30), which gives the relation

$$(2) \qquad \overline{P_1P_2}^2 = \overline{OP_1}^2 + \overline{OP_2}^2 - 2\,\overline{OP_1} \cdot \overline{OP_2} \cos P_2OP_1.$$

If, as in Figure 30,

$$(3) \qquad OP_1 = r_1, \quad OP_2 = r_2, \quad \angle P_2OP_1 = \theta_1 - \theta_2,$$

formula (1) readily follows from (2).

In all cases where both r_1 and r_2 are positive we have $\angle P_2OP_1 = \pm(\theta_1 - \theta_2) \pm n \cdot 360°$, where n is zero or a positive integer, so that we can always deduce (1) from (2). Formula (1) is true also when either r_1 or r_2 is negative, or when both are negative. This may be proved by examining the forms which equations (3) take in all possible cases.

Another method of proving (1) is to change from rectangular to polar coördinates in the distance formula of page 33, and to reduce the result to formula (1) by trigonometric processes. This is left as an exercise for the reader.

MISCELLANEOUS EXERCISES

1. Prove that the two triangles whose vertices are (x_1, y_1), (x_2, y_2), (x_3, y_3), and $(x_1 + a, y_1 + b)$, $(x_2 + a, y_2 + b)$, $(x_3 + a, y_3 + b)$, respectively, have their corresponding sides equal, and are congruent.

2. Prove that the two triangles whose vertices are (x_1, y_1), (x_2, y_2), (x_3, y_3), and $(x_1 - a, -y_1 + b)$, $(x_2 - a, -y_2 + b)$, $(x_3 - a, -y_3 + b)$, respectively, have their corresponding sides equal.

3. Prove that the points $(0, 0)$, $(2, 2)$, $(-2, 6)$, $(-4, 4)$ are vertices of a parallelogram by showing (*a*) that opposite sides are equal; (*b*) that opposite sides are parallel; (*c*) that the diagonals bisect each other.

4. Prove (*a*) by means of distances, (*b*) by means of slopes, that the points $A(1, -3)$, $B(-2, 3)$, $C(0, -1)$ lie on one straight line.

5. If A, B, C are as in Exercise 4, in what ratio does B divide AC? In what ratio does A divide BC?

6. Prove (*a*) by means of distances, (*b*) by means of slopes, that the points $(2a - b, c)$, $(b, 2a - c)$, (a, a) lie on one straight line.

7. If (x_1, y_1), (x_2, y_2), (x_3, y_3) are vertices of a triangle, prove that the three medians meet at the point whose coördinates are $(x_1 + x_2 + x_3)/3$, $(y_1 + y_2 + y_3)/3$.

Hint. Show that this point divides each median in the ratio 2 : 1.

8. The middle point of a segment is $(2, -3)$ and one end is $(0, -2)$. Find the coördinates of the other end.

9. Prove that $(-1, -2)$, $(6, 2)$, $(7, -1)$ are vertices of an isosceles triangle. Find the base angles.

10. If l_1 is the line on which $(a, 6)$, $(1, 2)$ lie and l_2 is the line on which $(1, 2)$, $(4, 8)$ lie, determine a so that the angle from l_1 to l_2 is 45°.

11. Three vertices of a parallelogram are $(0, 0)$, $(1, 2)$, $(2, 6)$. Find the coördinates of the three points each of which could be the fourth vertex.

12. The base angle α of an isosceles triangle is such that $\tan \alpha = \frac{3}{4}$, and the extremities of the base are $(-1, -2)$, $(3, 0)$. Find the coördinates of the two points each of which could be the vertex.

13. The line joining $A(-1, 4)$, $B(2, 0)$ and the line joining $C(1, 3)$, $D(-3, 0)$ meet at $E(a, b)$. Find a, b by using the equations

$$\text{slope of } AE = \text{slope of } AB, \quad \text{slope of } CE = \text{slope of } CD.$$

14. In what ratio is the segment from $(-1, -1)$ to $(4, 3)$ divided by the point at which it intersects the x-axis?

15. The slopes of two lines are 1 and 7 respectively. Find the slopes of the bisectors of the angles between those lines.

16. Find the center of the circle that passes through the points $(-4, -10)$, $(13, 7)$, $(-11, -3)$.

17. For the triangle whose vertices are $A(-1, 2)$, $B(3, -1)$, $C(2, 3)$, find the coördinates of the point D in which the side BC is met by a perpendicular to it through A.

Hint. BD has the same slope as BC, while AD is perpendicular to BC.

18. Prove that the area of the triangle $P_1P_2P_3$ in Figure 31 is

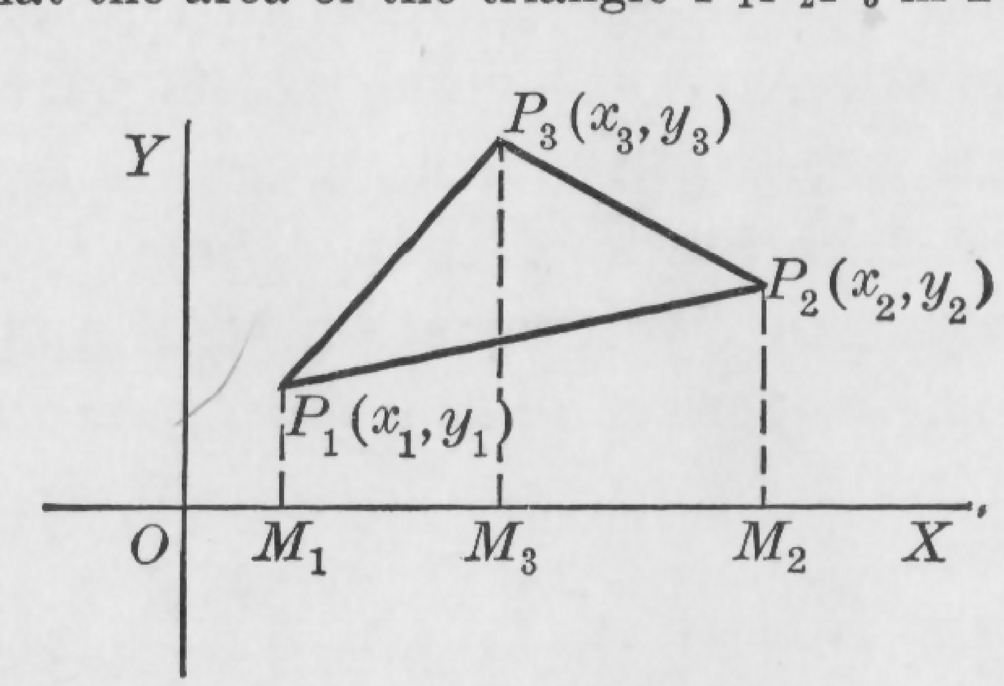


FIG. 31

$$\tfrac{1}{2}(x_1y_2 - x_2y_1 + x_2y_3 - x_3y_2 + x_3y_1 - x_1y_3) = \tfrac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}.$$

Hint. Area $P_1P_2P_3$ = Area $P_1P_3M_1M_3$ + Area $P_3P_2M_3M_2$ − Area $P_1P_2M_1M_2$. The area of a trapezoid is equal to one-half the sum of the parallel sides multiplied by the distance between those sides.

19. Find the formula for the area of triangle $P_1P_2P_3$ when P_3 is below P_1P_2 instead of above as in Figure 31. Prove that the result is the negative of that given in Exercise 18.

20. Find the area of the triangle whose vertices are $(0, -3)$, $(2, 1)$, $(4, 0)$.

CHAPTER III

EQUATIONS OF STRAIGHT LINES

19. Standard forms. An equation of a straight line is, by definition, an equation in x and y which is satisfied by the coördinates of every point on the line, and which is not satisfied by the coördinates of any point not on the line.

Whenever data are given which determine a line it is possible to find an equation of the line. In the present chapter we shall obtain, for example, an equation of a line through a given point with a given slope, an equation of a line through two given points, and an equation of a line with a given direction and at a given distance from the origin. When these data are expressed in general notation, the corresponding equations are said to be *standard forms.* It is possible to compare a given equation with a standard form and thus to find properties of the corresponding line, such as its direction, its distance from the origin, or where it cuts the axes.

It will be noted that all the standard forms which we shall derive in the following sections are of the first degree. This suggests the theorem, proved in § 25, that the locus of every equation of the first degree is a straight line.

20. Lines parallel to the axes. A line parallel to the y-axis and passing through the point $(a, 0)$ has the equation

$$x = a,$$

since this equation holds for every point on the line, and for no other points.

Similarly a line parallel to the x-axis and passing through the point $(0, b)$ has the equation

$$y = b.$$

The x-axis has the equation $y = 0$, and the y-axis has the equation $x = 0$.

21. Point slope form. *The line which has the slope* m *and which passes through the point* (x_1, y_1) *has the equation*

$$(1) \qquad y - y_1 = m(x - x_1).$$

To prove this theorem we first note that equation (1) is true if $x = x_1$, $y = y_1$. Next, let (x, y) be any point of the line other than (x_1, y_1). It follows from the theorem of page 36 that

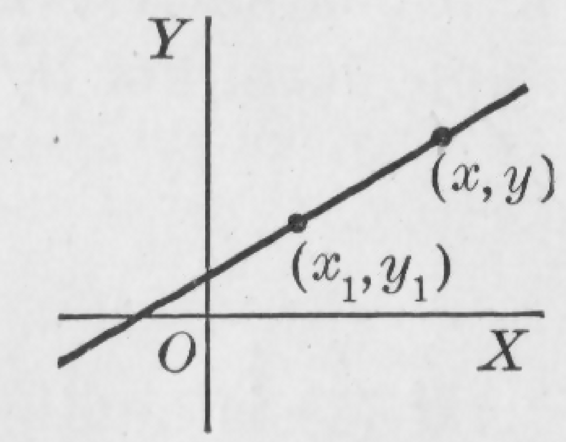


FIG. 32

$$(2) \qquad m = \frac{y - y_1}{x - x_1}.$$

If we multiply both sides of (2) by $(x - x_1)$ we obtain equation (1). Hence equation (1) holds for every point on the line.

On the other hand, if (x, y) is not on the line, equation (2), and therefore (1), cannot be true; for (2) states that the slope of the line passing through (x, y) and (x_1, y_1) is m, and this is true only of points (x, y) on the given line.

Thus (1) is true for every point on the line, and for no other points.

Any equation of the first degree that can be put into the form (1) is an equation of a line whose slope can be read off as m, and which passes through (x_1, y_1).

Example. — Show that $2x - y + 2 = 0$ is an equation of a straight line; find the slope of the line.

Solution. — The equation can be written

$$y - 0 = 2[x - (-1)],$$

hence this is an equation of a line of slope 2 passing through the point $(-1, 0)$.

22. Two point form. *The line passing through two points* (x_1, y_1), (x_2, y_2) *has the equation*

$$(1) \qquad y - y_1 = \frac{y_2 - y_1}{x_2 - x_1}(x - x_1).$$

Here we must except cases where $x_1 = x_2$; the line is then parallel to the y-axis, and its equation is $x = x_1$.

Equation (1) is only another way of writing the point slope form, for by the theorem of page 36, we have

$$m = \frac{y_2 - y_1}{x_2 - x_1}.$$

If (1) is multiplied through by $x_2 - x_1$ and the resulting equation is rearranged, we can express the equation in determinant notation (page 1) as follows

$$(2) \qquad \begin{vmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \end{vmatrix} = 0.$$

Equation (2) holds even when $x_1 = x_2$. By reference to Exercise 18, page 51, it will be seen that if (2) holds, then the triangle whose vertices are (x, y), (x_1, y_1), (x_2, y_2) is of area zero. We thus have another proof that if (x, y) satisfies (2), then the point is on the line passing through (x_1, y_1), (x_2, y_2).

Example. — Find an equation of the line which passes through the points $(-3, 2)$, $(2, -1)$.

Solution. — By using the two point form we have

$$y - 2 = \frac{-1-2}{2-(-3)}(x+3),$$
$$5y - 10 = -3x - 9,$$
$$3x + 5y - 1 = 0.$$

We could also have first computed the slope as $-3/5$, and have used the point slope form thus,

$$y - 2 = -\tfrac{3}{5}(x + 3).$$

We could then simplify the result as before.

EXERCISES

Draw a figure for every problem.

1. Find an equation of each of the lines all points of which are

(a) three units distant to the left from the y-axis;
(b) one unit distant from the x-axis and below that axis.

2. Find equations of lines all points of which are

(a) two units distant from the x-axis;
(b) two units distant from the y-axis.

3. Find an equation of each of the lines through a point given as follows, and with the slope specified:

(a) point $(0, 0)$, slope $= -1$;
(b) point $(1, 0)$, slope $= 2/3$;
(c) point $(0, -2)$, slope $= -5/4$;
(d) point $(-1, -3)$, slope $= 1/2$.

4. Find an equation of each of the lines through a point given as follows, and with the slope specified:

(a) point $(0, 0)$, slope $= -3/2$;
(b) point $(0, 2)$, slope $= 2$;
(c) point $(-3, 0)$, slope $= 2/5$;
(d) point $(-3, 2)$, slope $= -2/3$.

5. Find an equation of each of the lines through a point given as follows, and with inclination α as specified:

(a) point $(-1, -1)$, $\alpha = 45°$;
(b) point $(0, 2)$, $\alpha = 60°$;
(c) point $(-1, -1)$, $\alpha = 135°$;
(d) point $(0, 2)$, $\alpha = 120°$.

6. Find an equation of each of the lines through a point given as follows, and with inclination α as specified:

(a) point $(-2, 0)$, $\alpha = 45°$;
(b) point $(1, -1)$, $\alpha = 30°$;
(c) point $(-2, 0)$, $\alpha = 135°$;
(d) point $(1, -1)$, $\alpha = 150°$.

7. Find an equation of each of the lines through two points given as follows:

(a) $(-1, 0)$, $(2, 0)$; (b) $(1, 3)$, $(-1, -4)$;
(c) $(-5, 0)$, $(2, -3)$; (d) $(0, 1)$, $(0, -3)$.

8. Find an equation of each of the lines through two points given as follows:

(*a*) $(1, 0), (-3, 0)$; (*b*) $(1, -4), (-5, 2)$;
(*c*) $(1, 0), (-1, -1)$; (*d*) $(0, -2), (0, 2)$.

9. Find the slope of each of the lines which have the equations

(*a*) $x - 2y = 0$; (*b*) $2y + 3x - 1 = 0$;
(*c*) $x = y + 1$; (*d*) $y = 0$.

10. Find the slope of each of the lines which have the equations

(*a*) $2x + y = 0$; (*b*) $4x - y = 4$;
(*c*) $5x + 6y - 7 = 0$; (*d*) $\frac{x}{2} + \frac{y}{3} = 1$.

11. Find an equation of each of the lines through the point $(-2, 0)$ and

(*a*) parallel to the line $x - y = 4$;
(*b*) perpendicular to the line $x - y = 4$.

12. Find an equation of the perpendicular bisector of the segment whose end-points are $(0, 4)$ and $(2, 0)$.

13. Prove that the quadrilateral bounded by the lines

$$x - 2y = 0, \quad 2x + 3y = 0, \quad 2x = 4y + 3, \quad 6y = 7 - 4x$$

is a parallelogram.

14. Find equations of the two diagonals of the parallelogram given in Exercise 13.

15. Prove that the quadrilateral bounded by the lines

$$y + 3x = 0, \quad x - 3y + 2 = 0, \quad 5 - 2y - 6x = 0, \quad 3y = x - 2,$$

is a rectangle.

16. Find equations of the medians of the triangle whose vertices are $(-2, 3), (4, -1), (1, 0)$.

17. Find an equation of each of the lines through $(3, 0)$ and

(*a*) parallel to the line joining the points $(-2, 0), (0, -2)$;
(*b*) perpendicular to the latter line.

18. Find an equation of each of the lines through $(-1, -2)$ and

(*a*) parallel to the line joining the points $(-1, 3), (1, 2)$;
(*b*) perpendicular to the latter line.

23. Slope intercept form. The **intercept** of a line on the x-axis (sometimes called the ***x*-intercept**) is the abscissa of the point in which the line intersects that axis. Similarly the **intercept** on the y-axis (the ***y*-intercept**) is the ordinate of the point of intersection of the line and the y-axis.

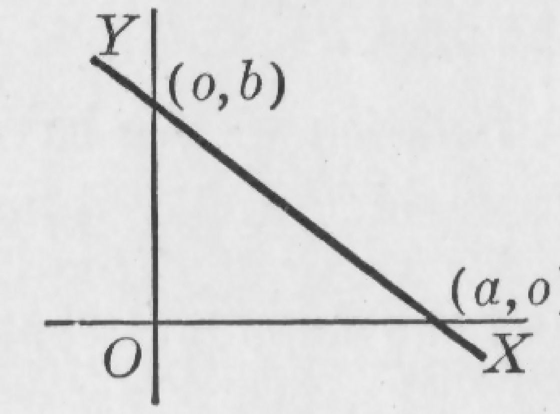


FIG. 33

The line whose slope is m *and whose intercept on the* y*-axis is* b *has the equation*

$$(1) \qquad y = mx + b.$$

This equation is at once deducible from the point slope form. To say that the intercept on the y-axis is b is equivalent to stating that the line passes through the point $(0, b)$. Its point slope equation is therefore

$$y - b = m(x - 0),$$

which is equivalent to (1).

It follows that if an equation can be put into form (1), it must represent a straight line of slope m and y-intercept b.

24. Intercept form. *The line whose intercept on the* x*-axis is* a, *and whose intercept on the* y*-axis is* b, *has the equation*

$$(1) \qquad \frac{x}{a} + \frac{y}{b} = 1,$$

if neither a *nor* b *is zero.*

We deduce (1) from the two point form, since the line must pass through the points $(a, 0)$, $(0, b)$. We thus have

$$y - 0 = \frac{b - 0}{0 - a}(x - a),$$

$$\frac{y}{b} = -\frac{x - a}{a},$$

$$\frac{x}{a} + \frac{y}{b} = 1.$$

proof

Every equation of form (1) has as its locus the line whose intercepts on the x- and y-axes are a and b respectively.

Example 1. — Find the slope intercept equation of the line whose inclination is 135°, and which intersects the y-axis in a point 2 units below the origin.

Solution. — We have here

$$m = \tan 135° = -1, \quad b = -2,$$

and the slope intercept equation of the line is

$$y = -x - 2.$$

Example 2. — Find the inclination and the intercepts on the axes of the line $x + 3y + 4 = 0$.

Solution. — By solving for y we put this equation in the slope intercept form

$$y = -\tfrac{1}{3}x - \tfrac{4}{3};$$

hence $m = -1/3$, $b = -4/3$. To find the inclination α such that $\tan \alpha = -1/3$, first find from Table III of page 11 the angle α' such that $\tan \alpha' = 1/3$. We find that $\alpha' = 17°$, approximately. Since $\alpha = 180° - \alpha'$, we have $\alpha = 163°$.

To put the given equation in intercept form we proceed as follows:

$$x + 3y + 4 = 0,$$
$$x + 3y = -4,$$
$$\frac{x}{-4} + \frac{3y}{-4} = 1,$$
$$\frac{x}{-4} + \frac{y}{-4/3} = 1.$$

Hence the intercepts on the axes are $a = -4$, $b = -4/3$. Note that a could also have been obtained by substituting $(a, 0)$ in the equation and solving the result. Similarly, substitution of $(0, b)$ would determine b.

25. Linear equations. Every straight line either has a slope and a y-intercept, in which case it has a point slope equation $y = mx + b$, or else it is parallel to the y-axis and has an equation $x = a$. Hence the following statement is true:

Every straight line has an equation of the first degree.

We now prove the converse theorem:

Every equation of the first degree has a straight line as its locus.

The general equation of the first degree is

$$(1) \qquad Ax + By + C = 0,$$

where A and B are not both zero. We first consider the case where B is zero, and then the case where B is not zero.

If B is zero then A is not zero, and equation (1) can be put in the form

$$x = -\frac{C}{A},$$

whose locus is a straight line parallel to the y-axis.

If B is not zero, we solve (1) for y and obtain the equivalent equation

$$y = -\frac{A}{B}x - \frac{C}{B}.$$

This is in the slope intercept form and its locus is a straight line of slope $-A/B$, and y-intercept $-C/B$.

Thus the locus of (1) is always a straight line. Though not originally for this reason, an equation of the first degree is often called a *linear equation.*

EXERCISES

Draw a figure for every problem.

1. Find the slope intercept equation of each of the lines whose slopes and y-intercepts are as follows:

(*a*) slope $= 0$, y-intercept $= -2$;
(*b*) slope $= -1$, y-intercept $= 0$;
(*c*) slope $= 2/3$, y-intercept $= -4/3$;
(*d*) slope $= -5$, y-intercept $= 5/2$.

2. Find the slope intercept equation of each of the lines whose inclinations, α, and y-intercepts, b, are as follows:

(*a*) $\alpha = 0$, $b = 0$; (*b*) $\alpha = 45°$, $b = 0$;
(*c*) $\alpha = 3\pi/4$ radians, $b = -2$; (*d*) $\alpha = 30°$, $b = 4/\sqrt{3}$.

3. Find the intercept equation of each of the lines whose intercepts are as follows, then express each equation in simplest form cleared of fractions:

(a) $a = -1,\ b = 2$; (b) $a = -2,\ b = -3$; (c) $a = 4,\ b = -4/5$.

4. Proceed as in Exercise 3 with the following data:

(a) $a = -2,\ b = 1$; (b) $a = 2/3,\ b = 2/5$; (c) $a = -3/4,\ b = -2/3$.

5. Find the slope and the y-intercept of each of the lines which have the equations

(a) $x - y = 1$; (b) $x + y + 1 = 0$;
(c) $2x - 2y = 3$; (d) $3x + 2y + 5 = 0$.

6. Find the slope and the y-intercept of each of the lines which have the equations

(a) $y + 2 = 0$; (b) $y + 2x - 2 = 0$;
(c) $3x + 2y = 0$; (d) $2x + 5y + 2 = 0$.

7. Find the x- and y-intercepts of the lines which have the equations given in Exercise 5.

8. Find the x- and y-intercepts of the lines which have the equations given in Exercise 6.

9. Obtain an equation of a line whose slope is m and whose x-intercept is a.

10. What is the x-intercept of a line whose slope is $-4/3$ and whose y-intercept is $2/3$?

11. Find an equation of a line through the point (2, 3) which makes equal intercepts on the x- and y-axes.

12. The point (4, 3) bisects the segment AB of a line whose intersection with the x-axis is the point A, and whose intersection with the y-axis is the point B. Show that the line has the equation $3x + 4y = 24$.

13. Find $\tan\theta$, where θ is the angle from the line which has the equation $x + 3y = 0$ to the line which has the equation $2x - y + 7 = 0$.

14. Find an equation of a line which is perpendicular to the line $y + x = 4$, and which has the y-intercept -4.

15. Find the slope of the line whose x- and y-intercepts are a and b respectively.

16. Find the x-intercept of a line whose slope is m and whose y-intercept is b.

26. Normal form. For the solution of problems involving the perpendicular distance from a line to a point (see pages 73–80) another standard form is convenient. This is the *normal form,* whose coefficients are expressed in terms of the length of the perpendicular from the origin to the line, and the angle from the x-axis to this perpendicular.

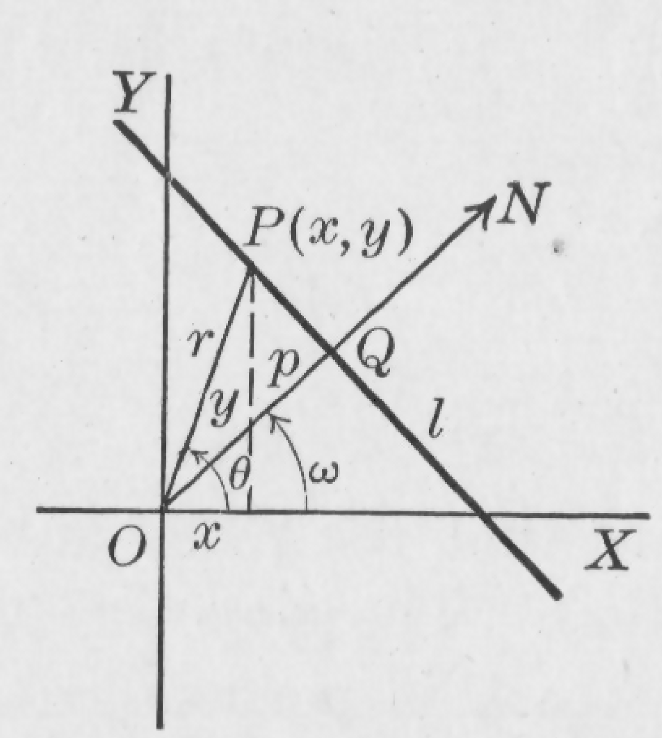


FIG. 34

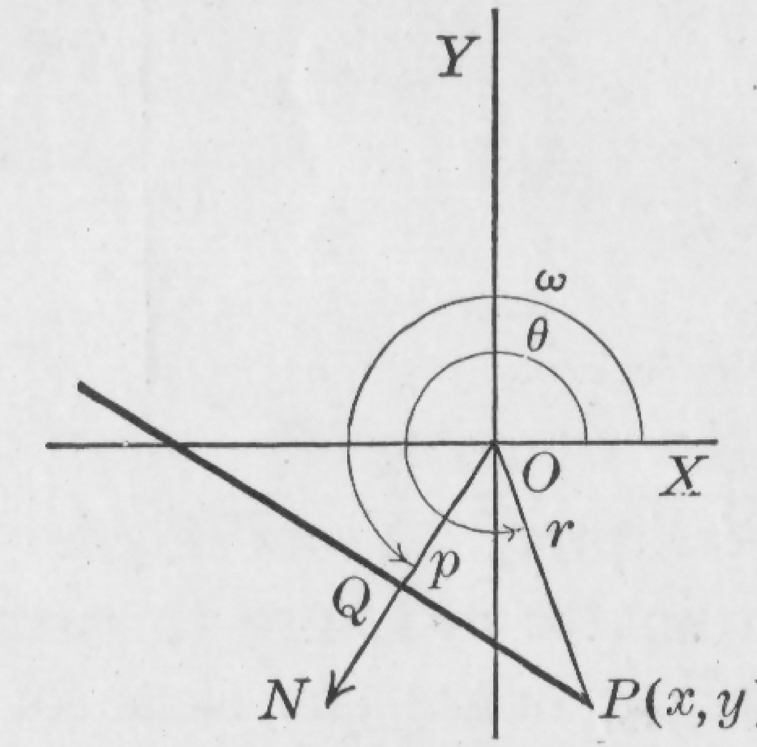


FIG. 35

Let the perpendicular ON intersect the line l at Q, and let the length of OQ be p. Let the angle from OX to ON be ω.* Here ω is the positive angle (less than 360°) from OX to ON, the latter line being directed according to the following rules: If the line l does not pass through the origin, then the positive direction of ON is from O toward Q; if l passes through the origin but does not coincide with the y-axis, we give ON the upward direction; if l is the y-axis, ON has the direction of OX. We call p the **normal intercept** and ω the **normal angle.** Note that there is one and only one line which has a given p and a given ω.

The straight line whose normal intercept is p *and whose normal angle is* ω *has the equation*

(1) $$x \cos\omega + y \sin\omega - p = 0.$$

We shall give two proofs of this statement. The first is in some ways the more simple of the two, but it assumes

* The Greek letter "omega."

that ω is not one of the angles 0°, 90°, 180°, 270°; a special proof would be required for each of these cases (see Exercise 17, page 68).

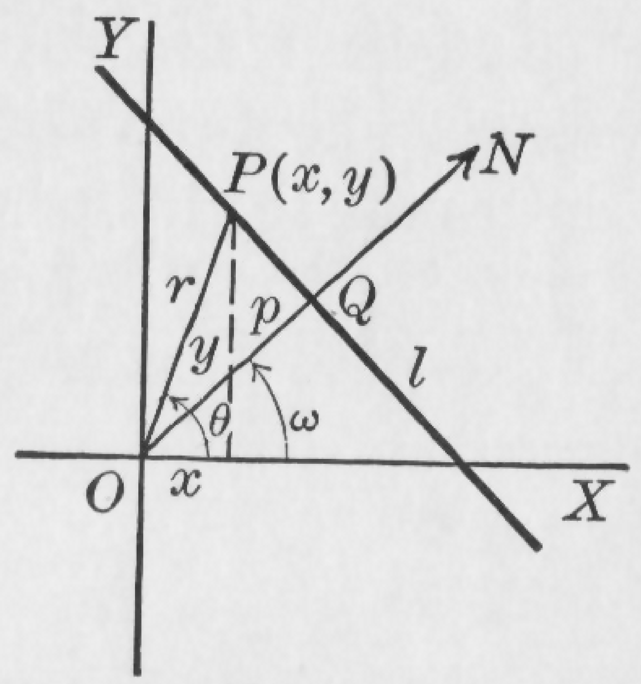


FIG. 34

First proof. From Figure 34 we see that the rectangular coördinates of Q are $(p \cos \omega, p \sin \omega)$. The slope of ON is $\tan \omega$; hence the slope of line l, which is perpendicular to ON, is $-1/\tan \omega$. The point slope equation of l is

$$y - p \sin \omega = \frac{-1}{\tan \omega}(x - p \cos \omega).$$

If we substitute $\sin \omega / \cos \omega$ for $\tan \omega$ and simplify the resulting equation, we obtain (1) by the following steps:

$$y - p \sin \omega = -\frac{\cos \omega}{\sin \omega}(x - p \cos \omega),$$

$$y \sin \omega - p \sin^2 \omega = -x \cos \omega + p \cos^2 \omega,$$

$$x \cos \omega + y \sin \omega - p(\sin^2 \omega + \cos^2 \omega) = 0,$$

$$x \cos \omega + y \sin \omega - p = 0.$$

Second proof. Let $P(x, y)$ be any point on the line l. Draw OP, and let polar coördinates of P be (r, θ), so that

(2) $$x = r \cos \theta, \qquad y = r \sin \theta.$$

The angle from OQ to OP is $\theta - \omega$. Since ON is perpendicular to the line l, we have $p/r = \cos(\theta - \omega)$, or

(3) $$p = r \cos(\theta - \omega).$$

Equation (3) holds for every line l, and for all positions of P on l. It is, moreover, true only when P is on l.

One of the addition formulas on page 8 gives

$$\cos(\theta - \omega) = \cos \theta \cos \omega + \sin \theta \sin \omega.$$

By using this relation we reduce (3) to the form

$$p = r \cos \theta \cos \omega + r \sin \theta \sin \omega.$$

With the aid of (2) this becomes, when we interchange sides of the equation,

$$x \cos \omega + y \sin \omega = p,$$

which is equivalent to (1).

27. Reduction of a linear equation to normal form. When an equation of a line is given in the form

(1) $$Ax + By + C = 0,$$

how shall we proceed in order to express p, the normal intercept, and ω, the normal angle of the line, in terms of A, B, and C? We shall answer this question by finding a constant k such that

(2) $$k(Ax + By + C) \equiv x \cos \omega + y \sin \omega - p.$$

When equation (1) has been multiplied through by k we say that it has been *reduced to normal form.*

Another way of stating the identity (2) is to write

(3) $$\begin{aligned} kA &= \cos \omega, \\ kB &= \sin \omega, \\ kC &= -p. \end{aligned}$$

In order to determine k, square and add the first two of equations (3). The result is

$$k^2A^2 + k^2B^2 = \cos^2 \omega + \sin^2 \omega = 1,$$

from which we see that k has one of the two values

(4) $$k = \pm \frac{1}{\sqrt{A^2 + B^2}}.$$

A closer examination of equations (3) will give us rules for the sign of k. We shall prove their correctness after first stating them in the following directions for reducing (1) to normal form.

To reduce the equation

$$Ax + By + C = 0$$

to normal form,

$$x \cos \omega + y \sin \omega - p = 0,$$

multiply each term by $\mathrm{k} = \pm 1/\sqrt{\mathrm{A}^2 + \mathrm{B}^2}$. *The following rules determine whether to take the + or the − sign in this formula for* k:

(*a*) *If* $\mathrm{C} \neq 0$, k *has the sign opposite to that of* C.

(*b*) *If* $\mathrm{C} = 0$ *and* $\mathrm{B} \neq 0$, k *has the same sign as that of* B.

(*c*) *If* $\mathrm{C} = 0$ *and* $\mathrm{B} = 0$, $\mathrm{k} = 1/\mathrm{A}$.

Rules (*a*), (*b*), and (*c*) follow from equations (3) and our conventions regarding ω. Thus the last of equations (3) shows that if C is not zero, then kC is negative, a statement from which rule (*a*) follows. Under the hypotheses stated in rule (*b*), the last of equations (3) tells us nothing about the sign of k, and we turn to the second of equations (3). Since $C = 0$, the line passes through the origin. We recall that for such a line we take the normal direction upwards, so that $\sin \omega$ is positive. Thus kB is positive, and rule (*b*) follows. In the case covered by rule (*c*) the line is $Ax = 0$ and its normal form is $x = 0$, hence the rule.

When k has been determined as above indicated, equations (3) express ω and p in terms of A, B, and C, and thus answer the question asked at the beginning of this section.

Example 1. — Write an equation of a line whose normal intercept is 5, and whose normal angle is 240°.

Solution. — Here $p = 5$, $\cos \omega = \cos 240° = -\cos 60° = -1/2$, $\sin \omega = -\sqrt{3}/2$, and the required equation is

$$x\left(-\frac{1}{2}\right) + y\left(-\frac{\sqrt{3}}{2}\right) - 5 = 0,$$

or

$$x + \sqrt{3}y + 10 = 0.$$

Example 2. — Find the normal intercept and the normal angle for the line which has the equation $2x - 4y - 1 = 0$.

Solution. — We put this equation in the normal form by multiplying through by $\pm 1/\sqrt{2^2 + (-4)^2} = \pm 1/2\sqrt{5}$; our rule requires that the multiplier be of sign opposite to that of -1. Hence the multiplier is $1/2\sqrt{5}$, and the normal form of our equation is

$$\frac{1}{\sqrt{5}}x - \frac{2}{\sqrt{5}}y - \frac{1}{2\sqrt{5}} = 0.$$

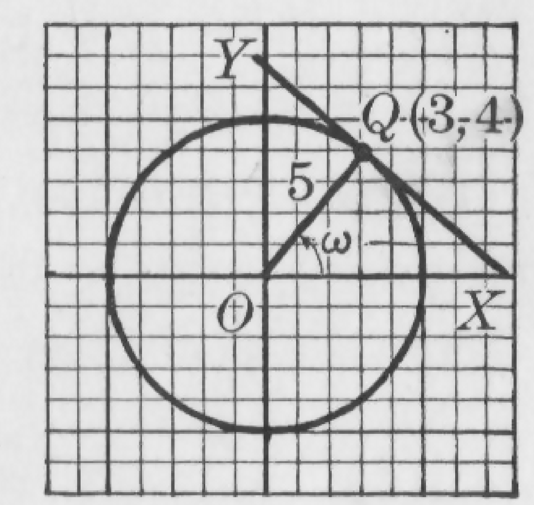
It follows that $\cos \omega = 1/\sqrt{5}$, $\sin \omega = -2/\sqrt{5}$, $p = 1/2\sqrt{5}$. The angle ω is in the fourth quadrant, and is $360° - \omega'$, where ω' is the angle in the first quadrant such that $\cos \omega' = \dfrac{1}{\sqrt{5}}$, $\sin \omega' = \dfrac{2}{\sqrt{5}}$, and hence $\tan \omega' = 2$. From the table of page 11 we have $\omega' = 63°$; hence $\omega = 297°$, approximately.

Example 3. — The point $Q(3, 4)$ is on a circle whose center is at the origin. Find the equation of the tangent to the circle at the point Q.

Solution. — We have

$$OQ = p = \sqrt{3^2 + 4^2} = 5,$$

$$\cos \omega = \frac{3}{5}, \quad \sin \omega = \frac{4}{5},$$



hence the normal form of the required equation is

$$\frac{3}{5}x + \frac{4}{5}y - 5 = 0,$$

which reduces to

$$3x + 4y - 25 = 0.$$

★ 28. Equivalent forms of equations. If one equation has been obtained from another by rearranging terms or by multiplying through by a constant other than zero, we have called the two equations *equivalent forms.* If an equation is of a certain degree, an equivalent form is of the same degree. The two equations have the same solutions; hence the geometrical locus of one equation is identical with that of the other.

A question involving the converse of the last statement of the preceding paragraph is this: Must two equations of the same geometric locus be equivalent? To show that the answer may be negative, consider the following three equations,

$$x(x^2 + y^2) = 0, \qquad x^3 = 0, \qquad x = 0.$$

Each is satisfied by points for which $x = 0$, and only by such points, but the three equations are not equivalent, though the first two are of the same degree. However, the answer would be affirmative if suitable restrictions were made on the type of equations considered. We shall not attempt a general discussion of this question; the following theorem regarding equations of straight lines will, however, give an answer for equations of the first degree.

All the equations of first degree of a given straight line must be equivalent.

It is in this sense that we may speak of *the* equation of a line. We now prove the theorem.

First, suppose the line is parallel to the y-axis; then it has an equation $x = k$. Any other first degree equation of the line can be written in the form $Ax + By + C = 0$. We are to show that the two equations

$$(1) \qquad x - k = 0, \qquad Ax + By + C = 0,$$

are equivalent.

Two points on the line are $(k, 0)$ and $(k, 1)$ since they satisfy the equation $x = k$. They must satisfy the second equation; hence we have

$$Ak + C = 0,$$
$$Ak + B + C = 0.$$

It follows that $C = -Ak$, and $B = 0$. Hence the equation $Ax + By + C = 0$ can be written $Ax - Ak = 0$, and is therefore the result of multiplying the equation $x - k = 0$ by A. This proves that the two equations (1) are equivalent.

Next suppose the line is not parallel to the y-axis; then it has a slope intercept equation $y = mx + b$. We are to show that any equation $Ax + By + C = 0$ of the line is equivalent to the slope intercept equation; that is, we are to prove the equivalence of the two equations of the same line

$$(2) \qquad y - mx - b = 0, \qquad Ax + By + C = 0.$$

From the first of these equations we see that the line passes through the points $(0, b)$ and $(1, m + b)$. Substitute these values for x and y in the second equation; we have

$$Bb + C = 0,$$
$$A + Bm + Bb + C = 0;$$

hence $C = -Bb$ and $A = -Bm$. The second of equations (2) can therefore be written $-Bmx + By - Bb = 0$, and is seen to be the result of multiplying $y - mx - b = 0$ by B. Thus equations (2) are equivalent.

In each case, we have shown that every other equation of first degree of the line is equivalent to a certain one; hence all linear equations of the line are equivalent.

EXERCISES

1. Write the equation of each of the straight lines whose normal angles and normal intercepts are as follows:

(*a*) $\omega = 0°$, $p = 0$; (*b*) $\omega = 270°$, $p = 2$;
(*c*) $\omega = 45°$, $p = 0$; (*d*) $\omega = 225°$, $p = 2$.

2. Proceed as in Exercise 1 with the following data:

(*a*) $\omega = 180°$, $p = 1$; (*b*) $\omega = 90°$, $p = 0$;
(*c*) $\omega = 30°$, $p = 10$; (*d*) $\omega = 150°$, $p = 4$.

3. Find the normal angle and the normal intercept of each of the lines which have the equations:

(*a*) $x = 0$; (*b*) $x - y = 0$;
(*c*) $x + y = 2\sqrt{2}$; (*d*) $y + \sqrt{2}x + \sqrt{3} = 0$.

4. Proceed as in Exercise 3 with the equations:

(*a*) $y = 0$; (*b*) $x + y = 0$;
(*c*) $x - y - \sqrt{2} = 0$; (*d*) $2x + 3y + 4 = 0$.

5. Find the perpendicular distance from the origin to each of the lines which have the equations:

(a) $x - y = 0$; (b) $3x + 4y = 5$;
(c) $x + y + \sqrt{2} = 0$; (d) $x - 2y = 4$.

6. Proceed as in Exercise 5 with the equations:

(a) $2x = 3y$; (b) $y + 5 = 0$;
(c) $y = 2x + \sqrt{5}$; (d) $2x + 4y + 5 = 0$.

7. The point (5, 12) is on a circle whose center is at the origin. Find an equation of the tangent to the circle at the point (5, 12).

8. Find equations of the two lines each of which has the slope $-1/2$ and the normal intercept 1.

9. Find the distance between the parallel lines which have the equations $3x + 4y = 5$, $3x + 4y - 10 = 0$.

10. Find the distance between the parallel lines which have the equations $5x + 12y = 10$, $5x + 12y + 3 = 0$.

11. Find the distance of a line from the origin when
(a) its intercepts are $a = 3$, $b = -4$;
(b) it passes through the points $(-1, 2)$, $(2, -1)$.

12. Find the distance of a line from the origin when
(a) its intercepts are $a = -12$, $b = 5$;
(b) it passes through the points $(-2, 0)$, $(3, 1)$.

13. Find the distance of a line from the origin when
(a) its slope is -2 and its y-intercept is -5;
(b) its slope is $2/3$, and it passes through the point $(-1, 2)$.

14. Find the distance of a line from the origin when
(a) its slope is $-1/2$, and its x-intercept is 5;
(b) its slope is $3/4$, and it passes through the point $(1, -2)$.

15. Derive the normal form for the equation of a line by expressing the intercepts a and b in terms of ω and p, and using the intercept form. In what cases must this proof be modified?

16. Derive the normal form for the equation of a line by expressing its slope and y-intercept in terms of ω and p, and using the slope intercept form. Are there cases which need another proof?

17. Finish the *first proof* of the normal form by taking up the cases $\omega = 0°$, $\omega = 90°$, $\omega = 180°$, $\omega = 270°$, and showing that formula (1) of page 61 holds true in each case.

POLAR COÖRDINATES

29. Equations of straight lines in polar coördinates. We have seen that every straight line has an equation of first degree in rectangular coördinates

$$Ax + By + C = 0.$$

We change to polar coördinates by substituting

$$x = r\cos\theta, \qquad y = r\sin\theta,$$

and obtain the equation

$$r(A\cos\theta + B\sin\theta) + C = 0. \tag{1}$$

If the line is parallel to the y-axis we have $B = 0$, $A \neq 0$, and (1) can be written

$$r\cos\theta = a. \tag{2}$$

Similarly a line parallel to the x-axis has the equation

$$r\sin\theta = b. \tag{3}$$

A line through the origin has the equation

$$\theta = \alpha, \tag{4}$$

where α is the inclination of the line.

Equation (3) of page 62 is the normal form in polar coördinates. Interchanging sides of the equation, we write

$$r\cos(\theta - \omega) = p. \tag{5}$$

Here ω is the normal angle, and p is the normal intercept.

EXERCISES

1. Obtain a polar equation of each of the lines satisfying the following conditions:
(a) parallel to the x-axis, y-intercept $= -1$;
(b) passing through the origin with slope $1/2$;
(c) normal angle $= 45°$, normal intercept $= \sqrt{2}$.

2. Proceed as in Exercise 1 with the following data:
(a) parallel to the y-axis, x-intercept $= 2$;
(b) coincident with the y-axis;
(c) normal angle $= 60°$, normal intercept $= 5$.

3. Change from rectangular to polar coördinates in each of the following equations:

(*a*) $x = 1$; (*b*) $x + y = 0$; (*c*) $3x + 4y - 5 = 0$.

4. Proceed as in Exercise 3 with the equations

(*a*) $y + 2 = 0$; (*b*) $x - 2y = 0$; (*c*) $x + y - 5\sqrt{2} = 0$.

5. Change from polar to rectangular coördinates and plot

(*a*) $r = 2 \csc \theta$; (*b*) $\theta = \pi/4$ radians; (*c*) $r \cos (\theta - 45°) = 5$.

6. Proceed as in Exercise 5 with the equations

(*a*) $r = 2 \sec \theta$; (*b*) $5 \sin \theta = 4$; (*c*) $r \cos (\theta - 60°) = 5$.

7. Find ω and p for the line which has the equation

$$r(\cos \theta + \sin \theta) = \sqrt{2}.$$

8. Find a polar equation of a line whose x- and y-intercepts are $a = -1$, $b = 2$.

9. Find a polar equation of the line passing through the points whose rectangular coördinates are $(1, 0)$, $(-2, -1)$.

10. Obtain a polar equation of the line passing through the points whose polar coördinates are $(1, 30°)$, $(2, 45°)$.

11. Find the slope of the line which has the polar equation $r \cos (\theta - 45°) = 2$.

12. Find the x- and y-intercepts of the line which has the polar equation $r \sin (\theta - 60°) = 3$.

MISCELLANEOUS EXERCISES

1. Find an equation of the perpendicular bisector of the segment whose ends are $(1, -2)$, $(3, 6)$.

2. Find an equation of a line through the point $(-2, 0)$ bisecting the segment whose ends are $(0, 2)$, $(3, -1)$.

3. In what ratio does the intersection point of the line whose equation is $x + 2y = 4$ and the segment from $(-1, -3)$ to $(9, 3)$ divide this segment?

4. Find the coördinates of the foot of the perpendicular from the origin to the line whose equation is $x + 2y = 5$.

5. Find the coördinates of the foot of the perpendicular from the point $(1, -1)$ to the line $3x + 4y - 24 = 0$.

6. Find an equation of each of the lines which pass through the intersection of the two lines $3x + y - 1 = 0$, $x + 2y + 3 = 0$ and which are (*a*) parallel, (*b*) perpendicular, to the line $3x + y = 7$.

7. Find an equation of a line which is perpendicular to the line $2x + 3y = 0$, and which has the x-intercept 10.

8. Find the x- and y-intercepts of a line whose normal intercept is 5, and which is parallel to the line $3x + 4y = 12$.

9. Find the slope of each of the lines which pass through the point $(5, -3)$, and whose normal intercept is 3.

10. Find an equation of each of the lines through the point $(3, 4)$ which have numerically equal x- and y-intercepts.

11. Find the point on the x-axis which is equidistant from the points $(2, 2)$, $(-1, 3)$.

12. Find the angle from the line $4x = 2y + 3$ to the line $3x + y = 6$.

13. Find an equation of each of the lines through the point $(1, -2)$ that make an angle of 45° with the line $3x + 2y + 1 = 0$.

14. The foot of the perpendicular from the origin to a line has the coördinates $(3, -4)$. Find an equation of the line.

15. A line has the normal intercept 4, and the product of its x- and y-intercepts is 32. Find an equation of the line (two solutions).

16. A line passes through the point $(7, 1)$ and is tangent to the circle of radius 5 whose center is at the origin. Find an equation of the line (two solutions).

17. The vertices of a triangle are $(-2, 0)$, $(3, 3)$, $(4, -2)$. Find

(*a*) equations of the sides;
(*b*) equations of lines through the vertices parallel to the opposite sides;
(*c*) equations of the perpendicular bisectors of the sides.

18. For the triangle of Exercise 17 find

(*a*) equations of the medians;
(*b*) equations of the lines through the vertices perpendicular to the opposite sides.

19. For the triangle whose sides are on the lines

$$3x - 2y = 1, \quad x + y - 2 = 0, \quad 2x - 3y - 4 = 0,$$

find

(*a*) equations of the medians;
(*b*) equations of lines through the vertices perpendicular to the opposite sides.

20. For the triangle of Exercise 19 find

(*a*) equations of lines through the vertices parallel to the opposite sides;

(*b*) equations of the perpendicular bisectors of the sides.

21. Prove that the locus of the equation

$$(A_1x + B_1y + C_1)(A_2x + B_2y + C_2) = 0$$

is the two lines whose equations are

$$A_1x + B_1y + C_1 = 0, \quad A_2x + B_2y + C_2 = 0.$$

22. Using the result of Exercise 21, plot the graphs of

(*a*) $x^2 - xy = 0$; (*b*) $2y^2 - 3xy - 2x^2 + x + 2y = 0$.

23. Proceed as in Exercise 22 with the equations

(*a*) $x^2 - y^2 + 2y - 1 = 0$; (*b*) $x^2 - xy - 6y^2 - 5y - 1 = 0$.

24. Find the perpendicular distance from the point $(1, -3)$ to the line which has the equation $3x - 4y = 10$.

Hint. Find the foot of the perpendicular from the point to the line.

25. Find the radius and the center of the circle which circumscribes the triangle formed by the lines

$$x + y = 7, \quad x = 3y + 3, \quad 2x - y + 4 = 0.$$

26. Prove that the area of the triangle whose vertices are $(0, 0)$, (x_1, y_1), (x_2, y_2) is equal to $\pm \frac{1}{2}(x_1y_2 - x_2y_1)$.

Hint. Find the base and the altitude.

27. Find the angles of the triangle whose sides are on the lines

$$y + 2x = 4, \quad y = 3x, \quad x - 2y = 2.$$

28. The sides of a parallelogram are on the lines

$$x + y = 4,\ x - 2y + 1 = 0,\ x + y + 2 = 0,\ x = 2y - 7.$$

Prove that a diagonal is trisected by the two lines joining the intersection of the first two sides with the mid-points of the other sides.

29. A vertex of a square is at $(2, -3)$ and the slope of the diagonal through that vertex is $-1/4$. Find equations of the lines on which lie the two sides through that vertex.

30. Two opposite vertices of a square are at $(-1, 1)$, $(3, 4)$. Find the other two vertices.

CHAPTER IV

PROBLEMS CONCERNING STRAIGHT LINES

30. Distance from a line to a point. In order to define the distance from a line l to a point $P_1(x_1, y_1)$ we drop a perpendicular from P_1 to l and designate the foot of this perpendicular by R; then $\overline{RP_1}$ will be the distance from l to P_1.

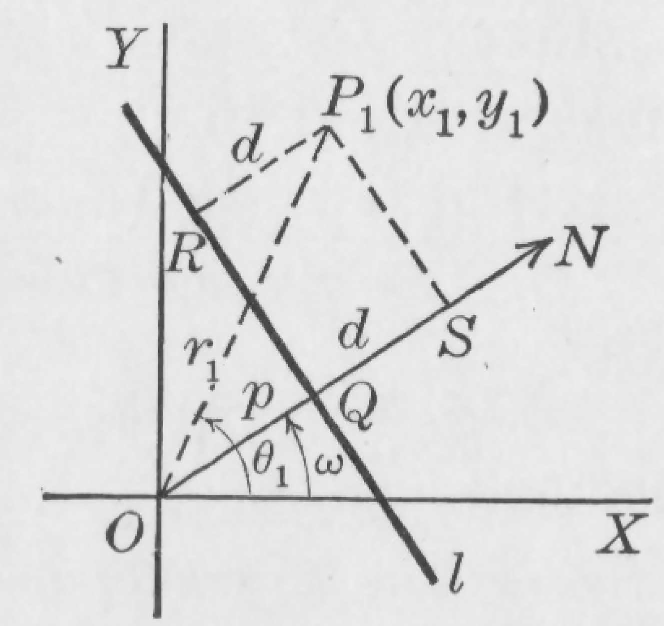


FIG. 36

The problem of finding the length of RP_1 is more easily solved if we treat RP_1 as a directed segment and designate its measure as the *directed distance d from the line l to the point* $P_1(x_1, y_1)$. The positive direction on RP_1 is taken the same as that of the perpendicular ON from the origin to the line (RP_1 and ON are parallel, since both are perpendicular to l). The positive direction of ON was defined on page 61 to be the direction from O to the line if the origin is not on the line, the upward direction for lines through the origin, and the direction OX for the line $x = 0$. In Figure 36, where P_1 and O are on opposite sides of l, the directed distance d is positive; if P_1 and O were on the same side of l, then d would be negative.

The following theorem gives a formula for d:

If the equation of l in normal form is

$$(1) \qquad x \cos \omega + y \sin \omega - p = 0,$$

then the directed distance d from the line l to the point $P_1(x_1, y_1)$ *is given by the formula*

$$(2) \qquad \boldsymbol{d = x_1 \cos \omega + y_1 \sin \omega - p.}$$

In other words, to find d, reduce the equation of l to normal form (1), then substitute x_1, y_1 for x, y in the left side of (1); the result will be equal to d.

To prove this formula, let us drop the perpendicular P_1S to ON. Then the right triangle OSP_1 will be a triangle of reference for the angle NOP_1 measured from ON to OP_1. If the length of OP_1 is r_1, and the angle XOP_1 is θ_1 (here r_1, θ_1 are polar coördinates of P_1), the angle NOP_1 is $\theta_1 - \omega$, and OS is $p + d$. The equation

$$(3) \qquad OS = OP_1 \cos NOP_1$$

is equivalent to

$$p + d = r_1 \cos(\theta_1 - \omega),$$
$$= r_1 \cos\theta_1 \cos\omega + r_1 \sin\theta_1 \sin\omega.$$

But

$$x_1 = r_1 \cos\theta_1, \quad y_1 = r_1 \sin\theta_1,$$

so that

$$p + d = x_1 \cos\omega + y_1 \sin\omega,$$

or

$$d = x_1 \cos\omega + y_1 \sin\omega - p.$$

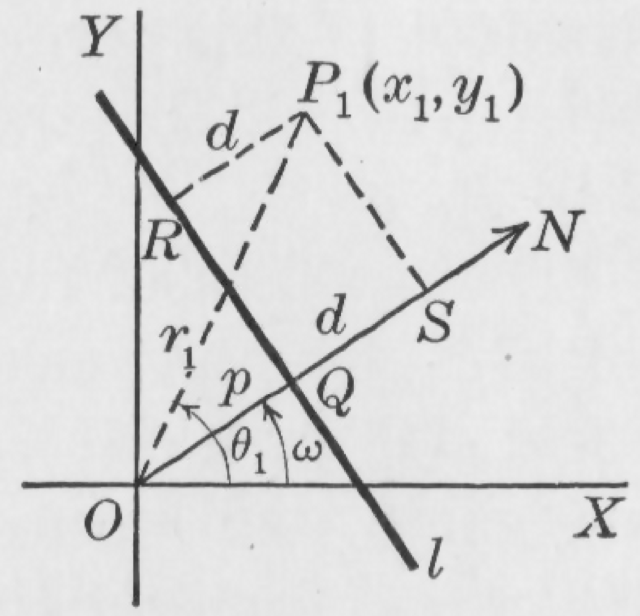


Fig. 36

The proof here given holds for all possible positions of P_1 and l; even if P_1 is on ON, so that OSP_1 is no longer a triangle, equation (3) is still true, and the correctness of formula (2) follows.

Note that if P_1 is a point (x, y) on l, then $d = 0$, and equation (1) holds. We thus have another proof for the normal form.

It follows from formula (2) and from § 27, pages 63, 64, that the distance from the line

$$Ax + By + C = 0$$

to the point $P_1(x_1, y_1)$ is given by the formula

$$(4) \qquad \boldsymbol{d = \frac{Ax_1 + By_1 + C}{\pm\sqrt{A^2 + B^2}}},$$

where the sign before the radical is opposite to that of C (if $C = 0$, see the rules of § 27).

Example. — Find the distance from the line $x + y - 2 = 0$ to the point $(-1, 2)$.

Solution. — By formula (4) we have

$$d = \frac{-1 + 2 - 2}{\sqrt{2}} = -\frac{1}{\sqrt{2}} = -\tfrac{1}{2}\sqrt{2}.$$

Thus the *undirected* distance from the line to the point is $\sqrt{2}/2$, and the point $(-1, 2)$ is on the same side of the line as the origin, since d is negative.

31. Bisector of an angle between two lines. In elementary geometry it is proved that the ray which bisects an angle is the locus of points equidistant from the sides of the angle. If the sides of the angle are each indefinitely prolonged through the vertex, then the angle and its vertical angle are bisected by one straight line. This is called the **bisector** of the angle. If we consider all the angles between two given straight lines, we find that there are two bisectors, which are perpendicular to each other.

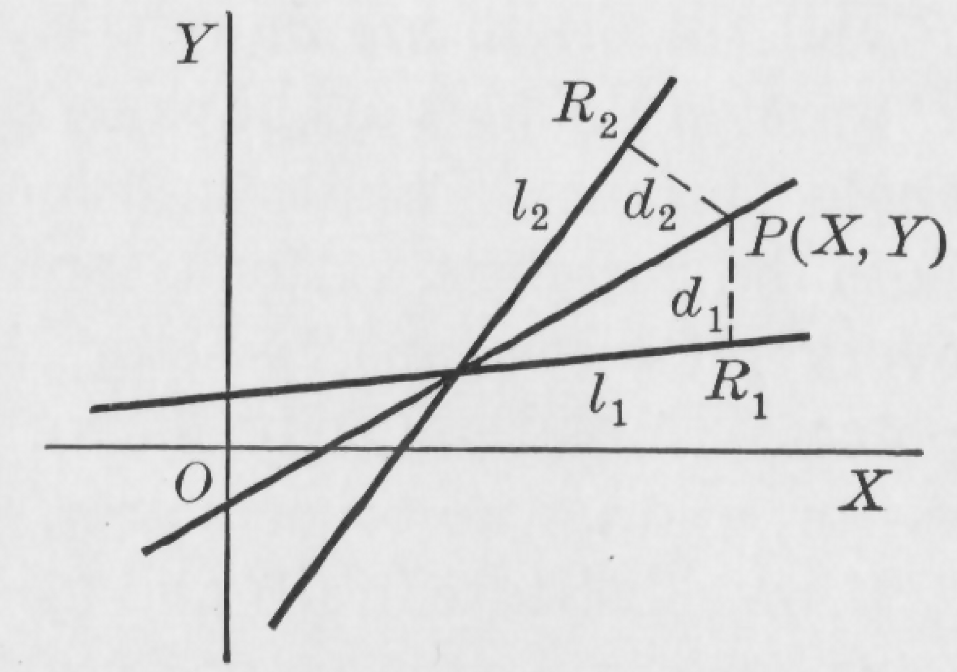


Fig. 37

To obtain an equation of the bisector of an angle between two straight lines, l_1, l_2, reduce the equation of each to normal form:

$$(1) \qquad \begin{aligned} l_1 &: \ x\cos\omega_1 + y\sin\omega_1 - p_1 = 0, \\ l_2 &: \ x\cos\omega_2 + y\sin\omega_2 - p_2 = 0. \end{aligned}$$

Let $P(X, Y)$ be a point on the bisector, and therefore equidistant from l_1 and l_2. Denote the *directed* distances from l_1 and l_2 to P, respectively, by d_1 and d_2. Then we have, from formula (2) of the preceding section,

$$(2) \qquad \begin{aligned} d_1 &= X\cos\omega_1 + Y\sin\omega_1 - p_1, \\ d_2 &= X\cos\omega_2 + Y\sin\omega_2 - p_2. \end{aligned}$$

The *undirected* distances from l_1 and from l_2 to P must be equal; these undirected distances are the numerical values of the directed distances d_1, d_2. If then, d_1 and d_2 are both positive, or both negative, we have

$$(3) \qquad d_1 = d_2;$$

but if one is positive and one negative, (3) is replaced by

$$(4) \qquad d_1 = -d_2.$$

In Figure 37, for example, d_1 and d_2 are both positive when P is any point on the bisector in the angle illustrated, since P and the origin are on opposite sides of both l_1 and l_2. If P were on the part of the same bisector that is in the vertical angle (the acute angle in which O lies), d_1 and d_2 would both be negative. Hence equation (3) remains true for every point on this bisector, and it is true for no other points; similarly (4) holds for all points on the other bisector, and for no other points.

If we substitute in (3) and (4) the values given by (2), and (since this can now be done without confusion of notation) replace X, Y by x, y in the resulting equations, we have the following equations of the two bisectors:

$$(5) \quad x \cos \omega_1 + y \sin \omega_1 - p_1 = x \cos \omega_2 + y \sin \omega_2 - p_2,$$

$$(6) \quad x \cos \omega_1 + y \sin \omega_1 - p_1 = -(x \cos \omega_2 + y \sin \omega_2 - p_2).$$

Formula (5) *gives an equation of the bisector of the angle within which* (*or its vertical angle*) *the origin lies; formula* (6) *gives an equation of the other bisector.*

The above rule applies to all cases except those where the origin is on one of the lines l_1, l_2. Here equations (3) and (4), expressed in terms of x and y, that is to say equations (5) and (6), are still the equations of the two bisectors. To tell which is an equation of the bisector of a given angle, take a point P on that bisector, and determine by the rule for the signs of d_1 and d_2 whether (3) or (4) is true; the corresponding equation (5) or (6) will be the desired equation.

Example 1. — Find the equations of the bisectors of the angles between the lines which have the equations

$$2x - 9y + 18 = 0, \quad 3x - 2y - 12 = 0.$$

Solution. — The normal forms of the above equations are

$$\frac{2x - 9y + 18}{-\sqrt{85}} = 0, \quad \frac{3x - 2y - 12}{\sqrt{13}} = 0.$$

Hence, from the preceding formulas (5) and (6), equations of the bisectors are

$$\frac{2x - 9y + 18}{-\sqrt{85}} = \frac{3x - 2y - 12}{\sqrt{13}},$$

$$\frac{2x - 9y + 18}{-\sqrt{85}} = -\frac{3x - 2y - 12}{\sqrt{13}}.$$

These equations are equivalent to

$$(2\sqrt{13} + 3\sqrt{85})x - (9\sqrt{13} + 2\sqrt{85})y + 18\sqrt{13} - 12\sqrt{85} = 0,$$

$$(2\sqrt{13} - 3\sqrt{85})x - (9\sqrt{13} - 2\sqrt{85})y + 18\sqrt{13} + 12\sqrt{85} = 0.$$

The former belongs to the bisector of the angle in which O lies.

Example 2. — Find the equation of the bisector of the acute angle between the y-axis and the line which has the equation $3x - 4y = 0$.

Solution. — The lines are as shown in Figure 38. Their equations in normal form are

$$x = 0, \quad \frac{3x - 4y}{-5} = 0.$$

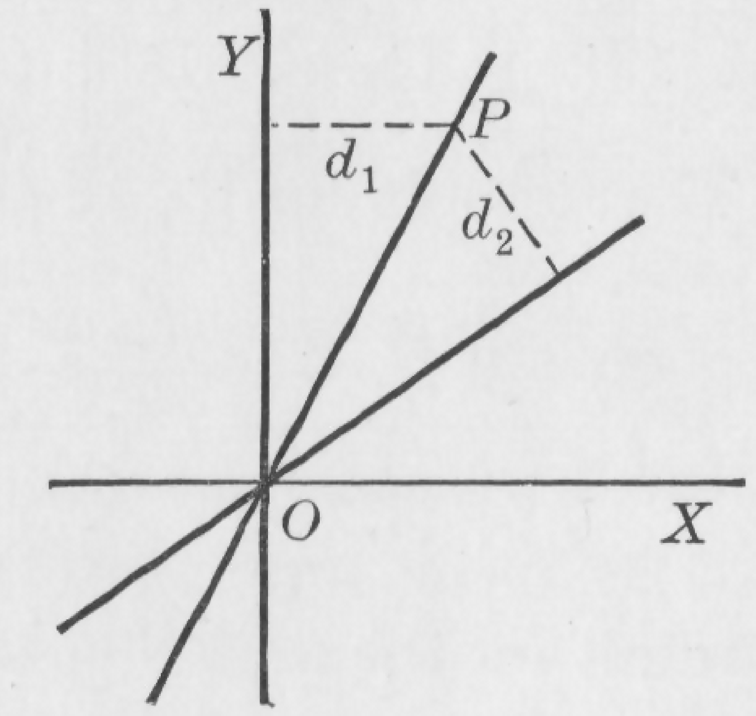


FIG. 38

The point P in Figure 38 is to the right of the line $x = 0$; hence by the rule of page 73, d_1 is positive. The other line passes through the origin, and, by the rule for such lines, d_2 is positive. Hence for this point, and so for all other points on the same bisector, we have $d_1 = d_2$. Thus equation (5) is the one to use here; the required equation is

$$x = \frac{3x - 4y}{-5},$$

which reduces to

$$2x - y = 0.$$

32. Area of a triangle. The area of a triangle whose vertices are $P_1(x_1, y_1)$, $P_2(x_2, y_2)$, $P_3(x_3, y_3)$ is equal to one half the product of $\overline{P_1P_2}$ and the corresponding altitude $\overline{MP_3}$. The length $\overline{P_1P_2}$ is given by the formula

$$(1) \qquad \overline{P_1P_2} = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}.$$

To obtain a formula for $\overline{MP_3}$ we proceed as follows: The two point equation of the line through P_1P_2 is

$$y - y_1 = \frac{y_2 - y_1}{x_2 - x_1}(x - x_1),$$

which becomes, after we multiply through by $x_2 - x_1$ and rearrange,

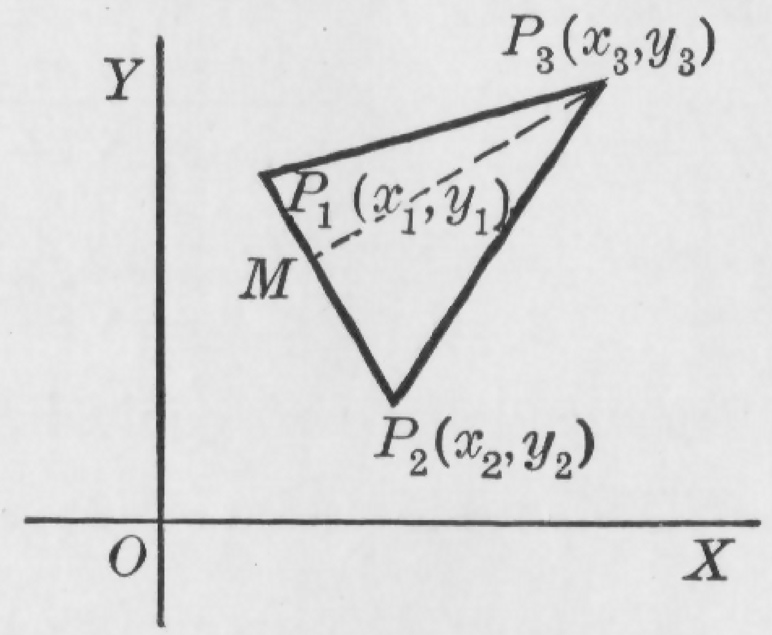


FIG. 39

$$(2) \qquad -(y_2 - y_1)x + (x_2 - x_1)y + x_1y_2 - x_2y_1 = 0.$$

The formula for the distance from this line to P_3 is

$$(3) \qquad \overline{MP_3} = \frac{-(y_2 - y_1)x_3 + (x_2 - x_1)y_3 + x_1y_2 - x_2y_1}{\pm\sqrt{(y_2 - y_1)^2 + (x_2 - x_1)^2}}.$$

By using formulas (1) and (3), we obtain the following result for the area S of the triangle $P_1\,P_2\,P_3$:

$$\begin{aligned} S &= \tfrac{1}{2}\overline{P_1P_2} \cdot \overline{MP_3} \\ &= \pm\tfrac{1}{2}[-(y_2 - y_1)x_3 + (x_2 - x_1)y_3 + x_1y_2 - x_2y_1]. \end{aligned}$$

By regrouping terms we obtain the equivalent formula

$$(4) \qquad S = \pm\tfrac{1}{2}[x_1(y_2 - y_3) - x_2(y_1 - y_3) + x_3(y_1 - y_2)].$$

Here the + or − sign is to be so taken that the right side of (4) is positive.

In determinant notation (see page 1), formula (4) becomes

$$(5) \qquad \textbf{Area of triangle } P_1P_2P_3 = \pm\tfrac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}.$$

EXERCISES

1. Find the distance (undirected) from the line to the point given as follows; in each case state whether point and origin are on the same side of the line:

(*a*) $4x + 3y = 10$, $(3, 3)$;
(*b*) $2x - 3y = 7$, $(-1, -1)$;
(*c*) $y = 4$, $(2, 3)$.

2. Proceed as in Exercise 1 for the following lines and points:

(*a*) $5x + 12y = 39$, $(5, 3)$;
(*b*) $x + y + 3 = 0$, $(-1, 2)$;
(*c*) $x + y = 0$, $(-2, -2)$.

3. Find the three altitudes of the triangle whose vertices are as follows: $(-1, 2)$, $(5, -2\frac{1}{2})$, $(-1, -7)$.

4. Find the three altitudes of the triangle whose vertices are $(0, 0)$, $(5, 1)$, $(-7, -4)$.

5. Find the equations of the bisectors of the angles between the lines whose equations are $3x - 4y + 5 = 0$, $5x + 12y = 60$.

6. Find the equations of the bisectors of the angles between the lines whose equations are $x + 5 = 0$, $8x - 15y = 17$.

7. Find the equation of the bisector of the angle from l_1 to l_2 if l_1 has the equation $x + y = 5$, and l_2 has the equation $2x + y + 4 = 0$.

8. Proceed as in Exercise 7 if l_1 has the equation $x + 4y = 5$, and l_2 has the equation $x - y = 0$.

9. Find the equations of the bisectors of the interior angles of the triangle whose vertices are given in Exercise 3.

10. Proceed as in Exercise 9 for the triangle given in Exercise 4.

Find the area of each triangle whose vertices are given in the following Exercises 11–16.

11. $(0, 0)$, $(1, 3)$, $(-1, 2)$.
12. $(0, -1)$, $(-4, 5)$, $(3, 2)$.
13. $(4, 0)$, $(2, 6)$, $(-2, 2)$.
14. $(-1, 1)$, $(2, -4)$, $(5, 0)$.
15. $(3, -1)$, $(-2, 4)$, $(5, 3)$.
16. $(-6, 0)$, $(-1, 7)$, $(-2, -2)$.

17. Find the area of the triangle formed by the lines

$$y = x, \quad y + x = 0, \quad x + 3y - 4 = 0.$$

18. Find the area of the triangle formed by the lines

$$5x + 3y + 6 = 0, \quad 2x - 3y - 6 = 0, \quad x + 2y - 3 = 0.$$

19. Prove analytically that the two bisectors of the angles between any two lines are perpendicular to each other.

20. Prove that if consecutive vertices of a convex quadrilateral are (x_1, y_1), (x_2, y_2), (x_3, y_3), (x_4, y_4), then the area of the quadrilateral is equal to

$$\pm \tfrac{1}{2}(x_1y_2 + x_2y_3 + x_3y_4 + x_4y_1 - x_2y_1 - x_3y_2 - x_4y_3 - x_1y_4).$$

33. Systems of straight lines. The equation $y = x + k$ contains, beside the variables x and y, the letter k to which different values may be given. For each value of k, the given equation represents a line. The totality of these lines constitutes a *system of lines*, and k is said to be the *parameter* of the system. All the lines of the system have the same slope, 1, and are thus parallel to each other.

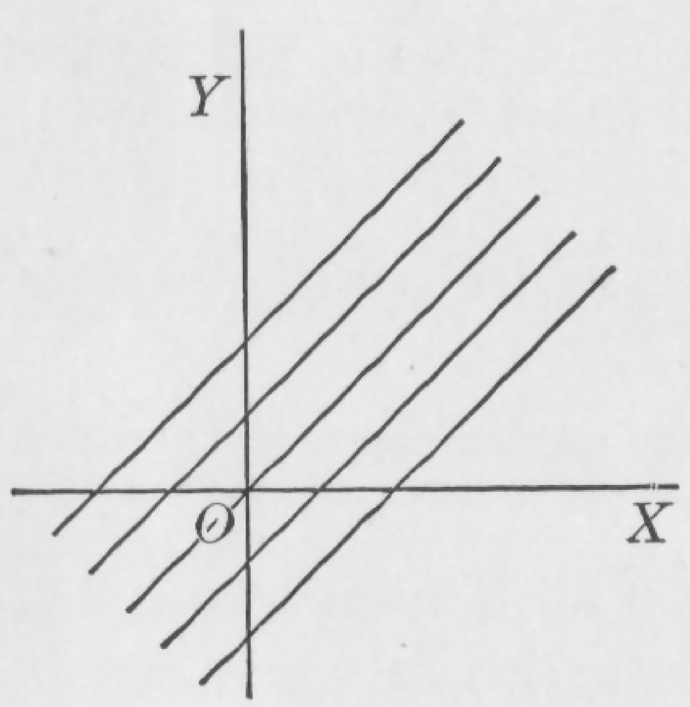


FIG. 40

The example given in the preceding paragraph suggests the following general definition:

If an equation of the first degree in x *and* y *contains, beside the variables* x *and* y, *a letter* k *to which different values may be given, the corresponding lines are said to form a* **system of lines** *and* k *is said to be the* **parameter** *of the system.*

The lines of a system have a common geometric property. Thus the equation $x \cos \omega + y \sin \omega - 10 = 0$ represents the system of lines, with parameter ω, all of which are at the distance 10 from the origin; that is, they are all tangent

to the circle whose center is at the origin and whose radius is 10.

As an application of the notion of a system of lines, consider the problem of finding the equation of a line satisfying two conditions which do not directly determine the coefficients of a standard form. We can solve such a problem by first writing an equation that represents the system of lines verifying one of the given conditions, and then determining the parameter (usually by reducing the equation to the appropriate standard form) so as to satisfy the other condition. This method is illustrated in the following Example 3.

Example 1. — Describe geometrically the system of lines which has the equation $x + By = 2$.

Solution. — If we put this equation in point slope form

$$y = -\frac{1}{B}(x - 2),$$

it is evident that all the lines for which B is not zero pass through the point $(2, 0)$; and the same is true when $B = 0$, since the corresponding equation is then $x = 2$. Therefore, all lines of the system pass through the point $(2, 0)$.*

Example 2. — Show that all lines parallel to a given line

$$(1) \qquad A_1x + B_1y + C_1 = 0$$

constitute a system which has the equation

$$(2) \qquad A_1x + B_1y + k = 0.$$

Solution. — If $B_1 = 0$, all lines parallel to (1) have equations of the form $x = k'$, which is equivalent to $A_1x = A_1k'$. We identify this last equation with (2) by writing $k = -A_1k'$ and recalling that B_1 is zero.

If B_1 is not zero, every line parallel to (1) has the slope $-A_1/B_1$, and therefore has a slope intercept equation

$$(3) \qquad y = -\frac{A_1}{B_1}x + b.$$

* The system does not, however, include *all* lines through $(2, 0)$. The line $y = 0$ is such a line which is not a member of the system. Similar exceptional cases often occur.

If we clear of fractions, rearrange terms, and make the substitution $B_1 b = -k$, equation (3) reduces to (2).

Example 3. — Find the equations of the lines which are parallel to $4x + 3y = 0$ and tangent to a circle of radius 10 whose center is at the origin.

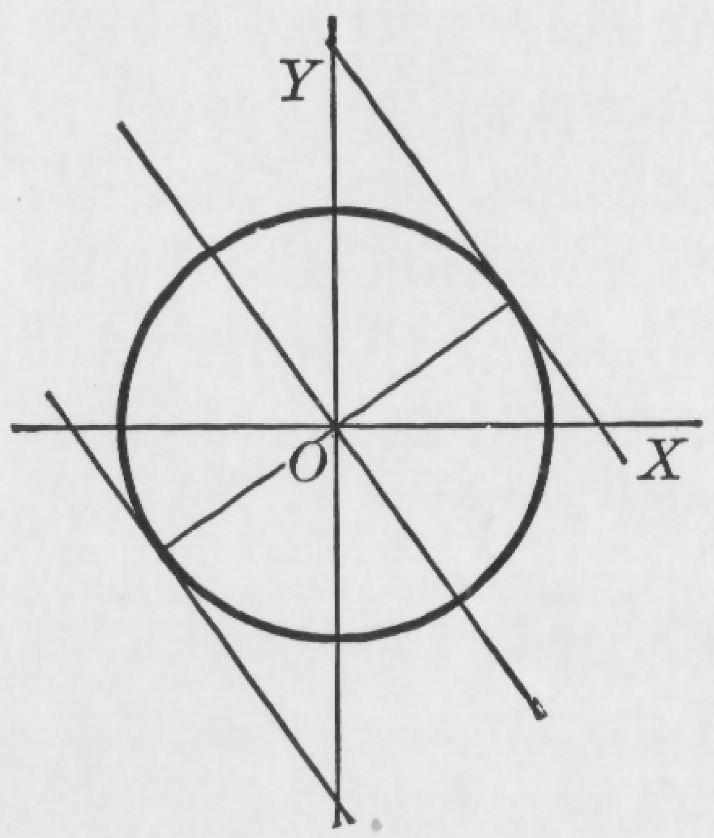


FIG. 41

Solution. — An equation of the system of lines parallel to $4x + 3y = 0$ is, by Example 2,

$$(4) \qquad 4x + 3y + k = 0.$$

The second of the given conditions is equivalent to the statement that the lines to be found have normal intercepts equal to 10. The normal form of lines (4) is

$$\frac{4}{\pm 5}x + \frac{3}{\pm 5}y - \frac{k}{\mp 5} = 0.$$

From this we see that the normal intercepts are $\mp k/5$. Hence for the lines to be found we have $\mp k/5 = 10$, or $k = \mp 50$. If we substitute these values for k in (4) we obtain the two required equations

$$4x + 3y - 50 = 0, \quad 4x + 3y + 50 = 0.$$

We could also have solved the problem by starting with the system of lines of normal intercept 10,

$$(5) \qquad x \cos \omega + y \sin \omega - 10 = 0,$$

and determining ω so that the slope in (5), $-\cot \omega$, is equal to the slope, $-4/3$, of the line $4x + 3y = 0$.

a, b, m, P, w

EXERCISES

Write an equation of each of the systems of lines described as follows.

1. Parallel to $x - y = 0$.
2. Parallel to $2x + 3y = 1$.
3. Passing through the origin.
4. Passing through the point (1, 0).
5. Having the normal angle 45°.
6. Having the normal intercept $p = 5$.

7. Perpendicular to $x - 2y + 1 = 0$.
8. Perpendicular to $A_1x + B_1y + C_1 = 0$.

For each of the following systems of lines name a property common to all the lines of that system.

9. $x + k = 0$. slope α 10. $\frac{x}{a} + y = 1$.

11. $kx + y - 2 = 0$. y intercept common   12. $x + y = k$.

13. $\frac{x}{k} - \frac{y}{k} = 1$.   14. $kx + \sqrt{1 - k^2}\,y = 5 \; (k^2 \leqq 1)$.

Find equations of the lines that satisfy the following conditions.

15. Inclination = 45°; x-intercept $= -4$.
16. The line passes through $(7, -1)$ and is tangent to the circle of radius 5 whose center is at the origin.
17. The line passes through (8, 2) and has equal intercepts on the axes.
18. The line has slope 4; sum of x- and y-intercepts is 3.
19. The line is 10 units distant from the origin, and is perpendicular to the line $2x - y = 4$. has slope $-\frac{1}{2}$
20. The line passes through $(4, -4)$ and forms with the coördinate axes in the first quadrant a triangle of area 4.

34. Lines through the point of intersection of two given lines.

Let the equations of two intersecting lines be

$$(1) \quad A_1x + B_1y + C_1 = 0, \qquad A_2x + B_2y + C_2 = 0.$$

Then the system of lines which has the equation

$$(2) \quad k_1(A_1x + B_1y + C_1) + k_2(A_2x + B_2y + C_2) = 0$$

is composed of all the lines which pass through the point of intersection of lines (1).

If in (2) we divide through by either k_1 or k_2 we see that this equation has essentially but one parameter, which may be taken as k_1/k_2 or k_2/k_1; hence (2) is the equation of a system of lines.

To show that every line of system (2) passes through (x_1, y_1), the point of intersection of lines (1), we observe that (x_1, y_1) satisfies each of equations (1) and hence, when substituted in (2) makes each expression in parentheses equal to zero; thus (x_1, y_1) satisfies (2).

To show that (2) represents *every* line through (x_1, y_1) we prove that k_1 and k_2 can be so chosen that (2) is satisfied by the coördinates of an arbitrarily chosen point (x_2, y_2) other than (x_1, y_1). This will be true if we choose (as is always possible) k_1 and k_2 so that they satisfy the equation

$$k_1(A_1x_2 + B_1y_2 + C_1) + k_2(A_2x_2 + B_2y_2 + C_2) = 0.$$

By means of formula (2) we can often solve problems concerning lines that intersect, without stopping to determine the coördinates of the points of intersection.

Example 1. — Find the equation of the line passing through the point $(-1, 2)$ and also through the point of intersection of the lines $x + y = 5$, $3x + y + 1 = 0$.

Solution. — The equation of the system of lines through the intersection of the given lines is

$$k_1(x + y - 5) + k_2(3x + y + 1) = 0. \tag{3}$$

Substitute $x = -1$, $y = 2$; the result is

$$k_1(-4) + k_2(0) = 0.$$

Hence we must take $k_1 = 0$, and we may take for k_2 any value except 0, for example $k_2 = 1$. With these values for k_1 and k_2 in (3) we obtain, as the solution of our problem, the equation $3x + y + 1 = 0$. Note that this is one of the given equations.

Example 2. — Find the equation of the line of slope $-3/4$, which passes through the intersection of the lines $x - 2y = 3$, $2x + y = 4$.

Solution. — The system of lines through the intersection of the given lines has the equation

$$k_1(x - 2y - 3) + k_2(2x + y - 4) = 0,$$

or, on rearranging terms,

$$(k_1 + 2k_2)x + (-2k_1 + k_2)y - 3k_1 - 4k_2 = 0. \tag{4}$$

By solving for y, we see that the slope of a line having equation (4) is

$$m = -\frac{k_1 + 2k_2}{-2k_1 + k_2}.$$

When this is set equal to $-3/4$, we have

$$-\frac{3}{4} = -\frac{k_1 + 2k_2}{-2k_1 + k_2},$$
$$6k_1 - 3k_2 = -4k_1 - 8k_2,$$
$$10k_1 = -5k_2.$$

Hence we can take $k_1 = -1$, $k_2 = 2$. If we substitute these values in (4) we have, as the solution of our problem,

$$3x + 4y - 5 = 0.$$

EXERCISES

1. Find the equation of the system, S, of lines through the intersection of $3x - y = 0$ and $y + 2x = 3$. Determine the equation

(*a*) of the line of S which passes through the point $(2, -1)$;
(*b*) of the line of S which has the slope -1.

2. Find the equation of the system, S, of lines through the intersection of $y = 2x + 5$ and $x - y = 4$. Determine the equation

(*a*) of the line of S which passes through the point $(2, 0)$;
(*b*) of the line of S which is parallel to $x - 3y = 4$.

3. Find the equation of the line parallel to the x-axis, and that of the line parallel to the y-axis, in the system of Exercise 1.

4. Find the equation of the line parallel to the x-axis, and that of the line parallel to the y-axis in the system of Exercise 2.

5. Given the triangle formed by the lines (*a*) $x + y - 3 = 0$, (*b*) $2x = y + 4$, (*c*) $y = x - 5$; without finding the coördinates of the vertices obtain the equations of the two lines, each of which passes through the vertex where (*a*) and (*b*) intersect and of which one is parallel to (*c*) and the other is perpendicular to (*c*).

6. Proceed as in Exercise 5 for the triangle formed by the lines (*a*) $y = 2x + 4$, (*b*) $x + 3y - 5 = 0$, (*c*) $x = y + 4$.

7. What does equation (2) of page 83 represent when the two lines (1) are parallel?

8. Show that if the two lines (1) of page 83 intersect, then no constants k_1, k_2, at least one of which is not zero, can be found which make the coefficients of both x and y in (2) equal to zero.

9. Show that if both the equations (1) of page 83 are in normal form, then equation (2) represents the locus of a point which moves so that the ratio of its distances from the lines (1) is equal to a ratio of the numerical values of k_2 and k_1.

10. Find an equation of the line which passes through the intersection of $2x - 3y + 5 = 0$ and $3x + 4y - 7 = 0$, and through the intersection of $x - 3y + 1 = 0$ and $4x + 4y - 3 = 0$, without finding these intersection points.

Hint. The line must be that one of the system through the first intersection point which has an equation equivalent to an equation of one of the system through the second intersection point.

★ 35. Relations between lines. Conditions that two lines be parallel, or be perpendicular to each other, have been given in terms of their slopes (page 40). It is useful also to express these relations in terms of the coefficients when the equations of the lines are of the general linear forms

$$(1) \qquad \begin{aligned} A_1x + B_1y + C_1 &= 0, \\ A_2x + B_2y + C_2 &= 0. \end{aligned}$$

I. *If there is a constant* k (*not equal to zero*) *such that*

$$(2) \qquad kA_1 = A_2, \quad kB_1 = B_2, \quad kC_1 \neq C_2,$$

then the two lines which have the equations (1) *are parallel; and conversely, if the two lines are parallel, there must be a* k *which satisfies the relations* (2).

Note that (2) is equivalent to the statement that the A's are in the same ratio as the B's, but not the same as the C's. If equations (2) are verified, the two equations (1) are not equivalent, and hence they represent non-coincident lines.

The first two of the relations (2) can be replaced by the equation

$$(3) \qquad A_1B_2 - A_2B_1 = 0.$$

To prove I, note that if equations (2) hold and if B_2 is not zero, then B_1 cannot be zero. The lines have slopes, $-A_1/B_1$ and $-A_2/B_2$ respectively, which are equal, since

$$-\frac{A_2}{B_2} = -\frac{k_1A_1}{k_1B_1} = -\frac{A_1}{B_1}.$$

Hence the lines are parallel. If B_2 were zero, B_1 would also be zero, and both lines would be parallel to the y-axis.

To prove the converse statement in I, suppose the lines are parallel. If they are not parallel to the y-axis they have slopes that are equal,

$$-\frac{A_1}{B_1} = -\frac{A_2}{B_2},$$

and neither B_1 nor B_2 vanishes. But the above equation can be written

$$\frac{B_2}{B_1}A_1 = A_2,$$

and obviously we have

$$\frac{B_2}{B_1}B_1 = B_2.$$

Hence if $k = B_2/B_1$, the first two of equations (2) are verified, while if the third relation (2) did not hold, the lines would be coincident. If the lines are parallel to the y-axis we have $B_1 = B_2 = 0$, and k is A_2/A_1.

II. *If the following relation is verified,*

$$(4) \qquad A_1A_2 + B_1B_2 = 0,$$

the two lines which have the equations (1) *are perpendicular, and conversely.*

To prove this theorem, first suppose B_1 and B_2 both different from zero; then, from (4), neither A_1 nor A_2 is zero. Write (4) in the form $A_1A_2 = -B_1B_2$ and divide both sides by B_1A_2; this gives

$$(5) \qquad \frac{A_1}{B_1} = -\frac{B_2}{A_2},$$

which shows that the slope of one line is the negative reciprocal of that of the other. Hence the lines are perpendicular.

If one of the B's, say B_1, is zero, then, from (4), $A_1A_2 = 0$, and since A_1 and B_1 are not both zero it follows that $A_2 = 0$. One line is then parallel to the x-axis, and the other to the y-axis; hence in this case also they are perpendicular to each other.

The converse theorem is proved by showing that if the lines are perpendicular to each other, but not parallel to the coördinate axes, then (5) holds, and therefore (4), while if the lines are parallel to the axes then either $A_1 = B_2 = 0$, or $A_2 = B_1 = 0$, and, in either case, (4) holds.

We can express the condition that three lines

$$(6)\qquad \begin{aligned} A_1x + B_1y + C_1 &= 0,\\ A_2x + B_2y + C_2 &= 0,\\ A_3x + B_3y + C_3 &= 0, \end{aligned}$$

be concurrent, that is, intersect in one point, as follows:

III. *If the three lines* (6) *are concurrent, then constants* k_1, k_2, k_3, *not all zero, can be found such that*

$$(7)\quad k_1(A_1x + B_1y + C_1) + k_2(A_2x + B_2y + C_2) + k_3(A_3x + B_3y + C_3) \equiv 0,$$

the identity holding for all values of x *and* y. *The converse is also true when no two of the three lines* (6) *are parallel.*

This theorem follows at once from the results of § 34. If the third line passes through the intersection of the first two, then some equation

$$k_1(A_1x + B_1y + C_1) + k_2(A_2x + B_2y + C_2) = 0$$

must be equivalent (see page 66) to

$$A_3x + B_3y + C_3 = 0.$$

It follows that the left side of the former of these last two equations is a constant times the left side of the latter. If this constant is called $-k_3$, the identity (7) is an immediate consequence.

To prove the converse statement, suppose the notation

so chosen that k_1 is not zero; then k_2 and k_3 are not both zero. Identity (7) is equivalent to

$$A_1x+B_1y+C_1 \equiv -\frac{k_2}{k_1}(A_2x+B_2y+C_2) - \frac{k_3}{k_1}(A_3x+B_3y+C_3).$$

From § 34 it follows that the first of lines (6) passes through the intersection of the other two. Hence if (7) is true, the lines (6) are concurrent.

If (7) is written

$$(8)\quad (k_1A_1 + k_2A_2 + k_3A_3)x + (k_1B_1 + k_2B_2 + k_3B_3)y + (k_1C_1 + k_2C_2 + k_3C_3) \equiv 0,$$

we see that (7) is true if and only if each expression in parentheses in (8) is equal to zero.

Another way of expressing the condition that the three lines (6) be concurrent is found if we solve the first two of equations (6) and substitute the solution for x and y in the third; the resulting expression must be zero. We shall not give the details of computation here; the result can be put in the form *

$$(9)\qquad \begin{vmatrix} A_1 & B_1 & C_1\\ A_2 & B_2 & C_2\\ A_3 & B_3 & C_3 \end{vmatrix} = 0.$$

Hence the rule:

If the three lines (6) *are concurrent, the determinant* (9) *must vanish; and conversely, unless the three lines* (6) *are parallel.*

Example. — Show that the following three lines are concurrent:

$$x - y + 1 = 0,\quad x + y - 3 = 0,\quad 3x - y - 1 = 0.$$

Solution. — We verify this statement by means of each of the relations (7) and (9).

By trying various numbers for the k's we may discover that we can take $k_1 = 2$, $k_2 = 1$, $k_3 = -1$, and that (7) then holds.

The test (9) is satisfied, since

$$\begin{vmatrix} 1 & -1 & 1\\ 1 & 1 & -3\\ 3 & -1 & -1 \end{vmatrix} = 0.$$

* This follows also from the theorem on page 3 regarding three simultaneous equations, if we take $z = 1$.

★ 36. Analytic solutions of geometric problems. The methods of this chapter enable us to prove analytically many propositions of geometry regarding the intersections of lines. We must be careful, in proving a proposition, to express our coördinates and equations in a form that will apply to *every* figure for which the proposition is to hold.

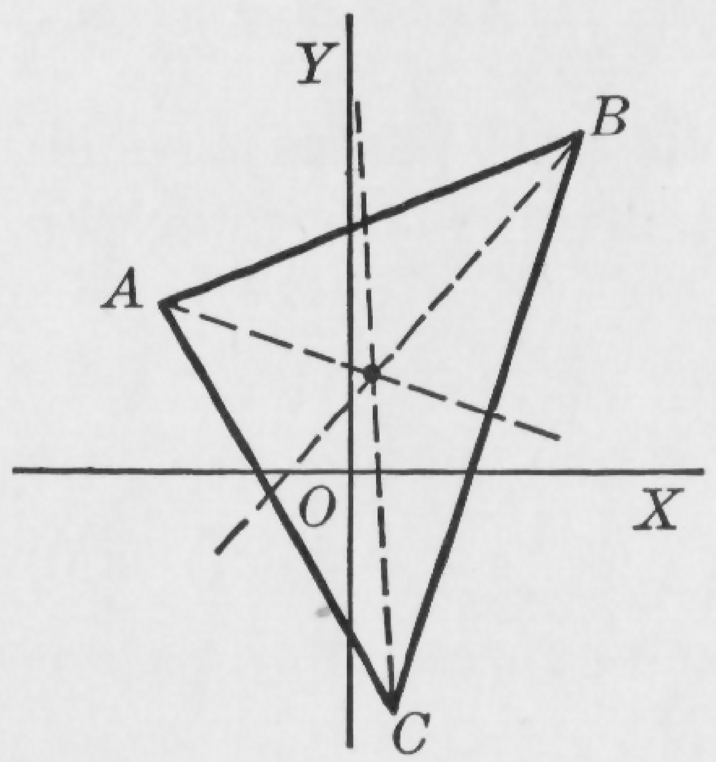


FIG. 42

Example 1. — Prove analytically that the three bisectors of the interior angles of a triangle are concurrent.

Solution. — Choose the axes so that the origin O is within the triangle, and let the equations of the three sides in normal form be briefly designated

$$L_1 = 0, \quad L_2 = 0, \quad L_3 = 0.$$

Then by the theorem of page 76, the equations of the three bisectors are

$$L_1 = L_2, \quad L_2 = L_3, \quad L_3 = L_1,$$

or

$$L_1 - L_2 = 0, \quad L_2 - L_3 = 0, \quad L_3 - L_1 = 0.$$

These lines are concurrent, according to § 35 (page 88) if k_1, k_2, k_3, not all zero, can be found such that

$$k_1(L_1 - L_2) + k_2(L_2 - L_3) + k_3(L_3 - L_1) \equiv 0.$$

Evidently this identity holds when $k_1 = k_2 = k_3 = 1$.

Example 2. — Prove analytically that the three perpendicular bisectors of the sides of a triangle are concurrent.

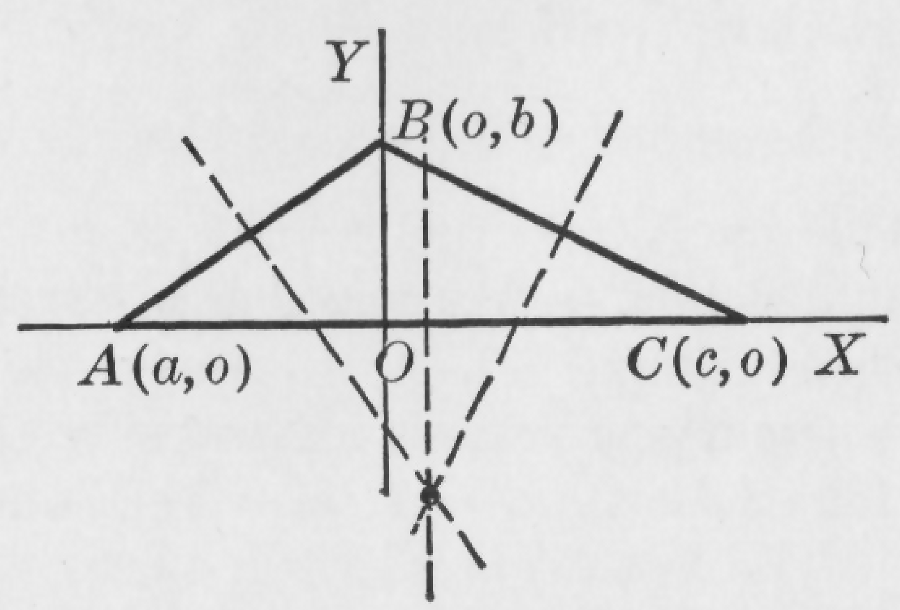


FIG. 43

Solution.—We give two proofs. In the first the coördinate axes are placed in a special position; in the other, the position of the axes does not matter.

For the first proof, take the axes and the coördinates of the vertices as in Figure 43; we can do so no matter what the shape or size of the triangle may be.

The perpendicular bisector of side AC has the equation

$$x - \frac{a+c}{2} = 0. \tag{1}$$

The perpendicular bisector of side BC passes through the mid-point of BC, whose coördinates are $(c/2, b/2)$, and its slope, the negative reciprocal of the slope of BC, is c/b. Its point slope equation is therefore

$$y - \frac{b}{2} = \frac{c}{b}\left(x - \frac{c}{2}\right),$$

which is equivalent to

$$cx - by + \frac{b^2 - c^2}{2} = 0. \tag{2}$$

Similarly, an equation of the perpendicular bisector of side AB is

$$ax - by + \frac{b^2 - a^2}{2} = 0. \tag{3}$$

The following identity, of type (7), page 88, which the student should verify, shows that (1), (2), and (3) are concurrent:

$$\left(cx - by + \frac{b^2 - c^2}{2}\right) - \left(ax - by + \frac{b^2 - a^2}{2}\right) + (a - c)\left(x - \frac{a+c}{2}\right) \equiv 0.$$

In the second proof we take the vertices as $A(x_1, y_1)$, $B(x_2, y_2)$, $C(x_3, y_3)$. Since the perpendicular bisector of AB is the locus of points (x, y) equidistant from A and B, it has the equation

$$\sqrt{(x - x_1)^2 + (y - y_1)^2} = \sqrt{(x - x_2)^2 + (y - y_2)^2},$$

which reduces to

$$[(x - x_1)^2 + (y - y_1)^2] - [(x - x_2)^2 + (y - y_2)^2] = 0. \tag{4}$$

This may be simplified and expressed as an equation of first degree, but such a transformation is not necessary. If we replace (x_1, y_1), (x_2, y_2) by (x_2, y_2), (x_3, y_3) respectively, we obtain an equation of the perpendicular bisector of side BC. We proceed similarly to derive an equation of the third bisector. The equations thus obtained are

$$[(x - x_2)^2 + (y - y_2)^2] - [(x - x_3)^2 + (y - y_3)^2] = 0, \tag{5}$$

$$[(x - x_3)^2 + (y - y_3)^2] - [(x - x_1)^2 + (y - y_1)^2] = 0. \tag{6}$$

If the left sides of (4), (5) and (6) are multiplied by 1 and added, everything cancels out, that is, the result is identically zero, which proves that the three perpendicular bisectors are concurrent.

EXERCISES

1. Apply theorems I and II of § 35 (pages 86, 87) to determine which pairs of the following lines are parallel, and which pairs are perpendicular:

$$(a)\ 2y = 3x - 5; \quad (b)\ 3x + 2y = 0; \quad (c)\ 2x = 3y - 4;$$
$$(d)\ 4x + 6y + 15 = 0; \quad (e)\ 6x - 9y = 20.$$

2. Proceed as in Exercise 1 with the following lines:

$$(a)\ y = 3x + 6; \quad (b)\ x = 3y + 6; \quad (c)\ 3x + y = 6;$$
$$(d)\ 3y - x + 3 = 0; \quad (e)\ y - 3x = 0.$$

3. Prove by one of the tests of § 35 that the following three lines are concurrent: $2x - y = 4$, $2y + x + 2 = 0$, $y = 2 - 3x$.

4. Prove by one of the tests of § 35 that the following three lines are concurrent: $y + 3x = 5$, $x = 2y$, $5x - 3y - 5 = 0$.

5. Prove that for every triangle the medians are concurrent.

6. Prove that for every triangle the three lines, each of which passes through a vertex and is perpendicular to the opposite side, are concurrent.

7. Prove that for every triangle ABC the bisector of the interior angle at A and the bisectors of the exterior angles at B and C are concurrent.

8. Prove that for every triangle ABC the bisector of the interior angle at A divides the opposite side BC into segments whose ratio is equal to $\overline{AB} : \overline{AC}$.

MISCELLANEOUS EXERCISES

For each of the triangles ABC *whose vertices are given as follows, find* (a) *an equation of the line through* A *parallel to* BC; (b) *an equation of the line through* A *perpendicular to* BC; (c) *the length of the altitude when* BC *is taken as base.*

1. $A(0, 0)$, $B(-2, 2)$, $C(2, 4)$.

2. $A(1, 4)$, $B(-1, -1)$, $C(3, -1)$.

3. $A(-3, 2)$, $B(1, 5)$, $C(0, -4)$.

4. $A(1, -1)$, $B(-4, 1)$, $C(4, 2)$.

For each of the triangles ABC *given as follows, find* (a) *tan* A *(the tangent of the interior angle at* A*);* (b) *the equation of the bisector of that angle;* (c) *the area of the triangle.*

5. As in Exercise 1. **6.** As in Exercise 2.

7. As in Exercise 3. **8.** As in Exercise 4.

For each of the triangles formed by the lines whose equations are given as follows, find (a) *the equation of the bisector of the interior angle formed by the first two lines;* (b) *the equation of the line through the intersection of the first two lines, and perpendicular to the third line;* (c) *the area.*

9. $2x + y - 18 = 0$, $x + 3y - 14 = 0$, $x - 7y + 36 = 0$.

10. $2x + y - 8 = 0$, $3x - y - 12 = 0$, $x + 3y - 14 = 0$.

11. $y = 3x + 2$, $x + 2y + 9 = 0$, $x + y + 2 = 0$.

12. $x + 3y = 6$, $5x = y - 2$, $7y = 3(x - 6)$.

Solve the following problems.

13. Find equations of the bisectors of the interior angles of the triangle whose vertices are $(0, 0)$, $(0, \sqrt{2})$, $(\sqrt{2}, 0)$, and prove that these bisectors are concurrent.

14. Find equations of the medians of the triangle of Exercise 10, and prove that these medians are concurrent.

15. Find equations of the three perpendicular bisectors of the sides of the triangle of Exercise 12, and prove that these bisectors are concurrent.

16. Find equations of the three lines, each of which passes through a vertex of the triangle of Exercise 11 and is perpendicular to the opposite side. Prove that these three lines are concurrent.

17. Find the vertices and the area of the triangle each of whose sides is parallel to a side of the triangle of Exercise 9 and passes through the opposite vertex.

18. Find the area of the quadrilateral whose consecutive vertices are $(-4, 1)$, $(-3, -2)$, $(1, -1)$, $(0, 2)$.

19. Find the area of the convex quadrilateral formed by the lines $x = 0$, $3x + y = 15$, $x - 6y + 14 = 0$, $x + y = 0$.

20. Find equations of the two lines through $(0, 2)$ which are equidistant from the points $(1, -1)$, $(4, -3)$.

21. Prove that if $P(x_1, y_1)$ lies above the line $Ax + By + C = 0$, where B is positive, then $Ax_1 + By_1 + C > 0$, and conversely. Use this theorem to determine whether the point $(4, -3)$ and the point $(-2, -1)$ are on the same side, or on opposite sides, of the line $x + 3y + 4 = 0$.

22. State and prove a theorem similar to that of Exercise 21, which will determine whether $P(x_1, y_1)$ lies to the right or left of the line $Ax + By + C = 0$, where A is positive. Use this theorem to solve the problem in the last sentence of Exercise 21.

23. Find the equations of each of the common tangents of the two circles C_1, C_2, where C_1 is of radius 2 and has its center at $(0, 0)$, while C_2 has its center at $(7, 1)$ and is of radius 3.

$\frac{d^2}{4} + \frac{e^2}{4}$

CHAPTER V

THE CIRCLE

37. The standard form of the equation of a circle. We recall that a circle is a plane figure defined as the locus of points at a given distance, the radius, from a given point, the center. To obtain an equation for a circle, take a set of rectangular coördinates in the plane of the circle, let $C(h, k)$ be the center and let R be the radius. If $P(x, y)$ is any point on the circle, then

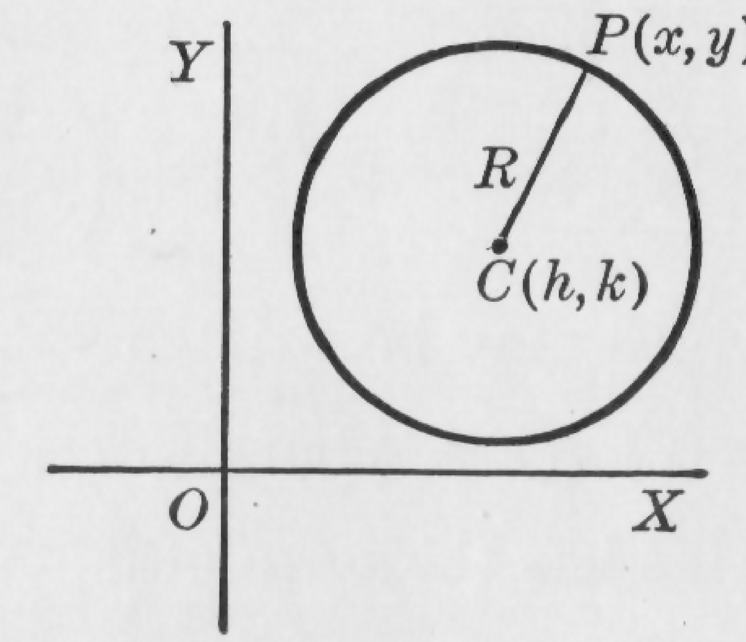


FIG. 44

$$(1) \qquad CP = R.$$

From the formula for the distance between two points we have

$$(2) \qquad (x - h)^2 + (y - k)^2 = R^2.$$

Conversely if equation (2) is satisfied, so also is equation (1), and P must lie on the circle. Thus equation (2) is satisfied by the coördinates of every point on the circle and by those of no other point. It is therefore an equation of the circle, called the *standard form.*

If the center of the circle is at the origin of coördinates, we have $h = 0$, $k = 0$, and hence

$$(3) \qquad x^2 + y^2 = R^2.$$

38. Equations reducible to the standard form. If equation (2), § 37, is expanded and rearranged it becomes

$$x^2 + y^2 - 2hx - 2ky + h^2 + k^2 - R^2 = 0.$$

This is of the form

$$(1) \qquad \boldsymbol{x^2 + y^2 + Dx + Ey + F = 0,}$$

where D, E, and F are constants.

The questions arise: 1. Can every equation of the form (1) be reduced to the standard form of equation (2), § 37? 2. Is every such equation an equation of a circle?

To answer these questions, we first rewrite equation (1) thus:

$$(x^2 + Dx) + (y^2 + Ey) = -F.$$

We then add $(\frac{1}{2}D)^2$ and $(\frac{1}{2}E)^2$ to both sides of the equation, and obtain

$$\left(x^2 + Dx + \frac{D^2}{4}\right) + \left(y^2 + Ey + \frac{E^2}{4}\right) = \frac{D^2}{4} + \frac{E^2}{4} - F,$$

which may be written

$$(2) \quad (x + \tfrac{1}{2}D)^2 + (y + \tfrac{1}{2}E)^2 = \tfrac{1}{4}(D^2 + E^2 - 4F).$$

This has the form of equation (2), § 37, where

$$(3) \quad \boldsymbol{h = -\tfrac{1}{2}D, \quad k = -\tfrac{1}{2}E, \quad R = \tfrac{1}{2}\sqrt{D^2 + E^2 - 4F}.}$$

Our first question is answered affirmatively.

The answer to our second question, however, is negative. To see this we note that in equation (2), which is equivalent to (1), the right member may be negative, while the left member cannot be less than zero for any point (x, y). In such a case the equation has no locus.

By inspection of equation (2) we arrive at the following conclusions concerning the locus of equation (1):

(*a*) *If* $D^2 + E^2 - 4F > 0$, *the locus of* (1) *is a circle with center* $(-\frac{1}{2}D, -\frac{1}{2}E)$ *and radius* $\frac{1}{2}\sqrt{D^2 + E^2 - 4F}$.

(*b*) *If* $D^2 + E^2 - 4F = 0$, *the locus of* (1) *is the point* $(-\frac{1}{2}D, -\frac{1}{2}E)$ (*sometimes called, in this connection, a* ***point circle***).

(*c*) *If* $D^2 + E^2 - 4F < 0$, *the equation* (1) *has no locus* (*it is sometimes said that the locus is an* ***imaginary circle***).

Example. — Find the locus of the equation

$$x^2 + y^2 + 6x - 8y + 9 = 0.$$

First solution (using the formulas). — Here $D = 6$, $E = -8$, $F = 9$ and hence $D^2 + E^2 - 4F = 64$. The locus is therefore a circle with center at $(-3, 4)$ and radius 4.

Second solution (by completing squares). — Collecting terms we have

$$(x^2 + 6x) + (y^2 - 8y) = -9.$$

If we complete the squares, we obtain

$$(x^2 + 6x + 9) + (y^2 - 8y + 16) = -9 + 9 + 16$$

or

$$(x + 3)^2 + (y - 4)^2 = 16.$$

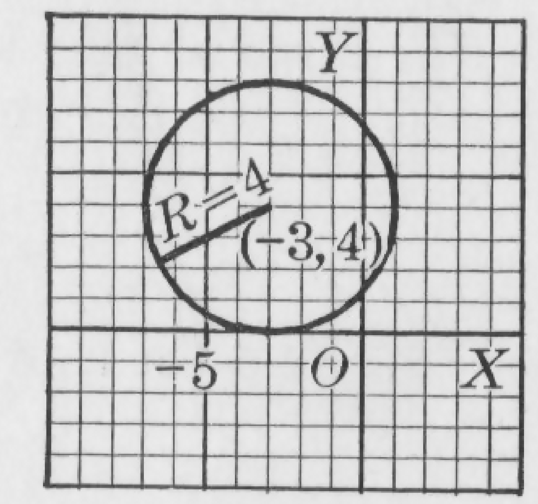


FIG. 45

Comparing with equation (2), § 37, we observe that the locus is a circle for which $h = -3$, $k = 4$, $R = 4$.

The method of the second solution is employed for curves other than circles, and is to be used in exercises of the following set.

EXERCISES

1. Find an equation of the circle:

(*a*) whose center is (0, 4) and whose radius is 6;
(*b*) whose center is (6, − 2) and whose diameter is 10;
(*c*) whose center is (− 3, 4) and which passes through (2, 5).

2. Find an equation of the circle:

(*a*) whose center is (6, 0) and whose radius is 4;
(*b*) whose center is (− 4, 4) and whose diameter is 7;
(*c*) whose center is (− 5, 0) and which passes through (1, 0).

3. Find the center and radius of each of the following circles which is real, and draw figures:

(*a*) $x^2 + y^2 - 8x = 0$;
(*b*) $x^2 + y^2 + 4x - 8y - 5 = 0$;
(*c*) $x^2 + y^2 - 6x + 8y + 25 = 0$;
(*d*) $x^2 + y^2 + 4x + 8 = 0$;
(*e*) $3x^2 + 3y^2 + 5x + 12y = 0$.

4. Proceed as in Exercise 3 with each of the following circles:

(*a*) $x^2 + y^2 - 36 = 0$;
(*b*) $x^2 + y^2 - 36y = 0$;

(*c*) $x^2 + y^2 + 8x + 4y + 20 = 0$;
(*d*) $x^2 + y^2 + 6x + 10 = 0$;
(*e*) $5x^2 + 5y^2 - 15x + 8y + 12 = 0$.

5. The extremities of a diameter of a circle are the points $A(4, -2)$ and $B(-2, 6)$. Find an equation of the circle.

6. The line joining $A(-2, 4)$ and $B(6, -2)$ is a diameter of a circle. Find an equation of the circle.

7. Find an equation of the straight line which passes through the centers of the two circles

$$x^2 + y^2 + 10x + 12y = 0,$$
$$x^2 + y^2 - 6x + 8y = 0.$$

8. Find the shortest distance from the point $(-2, 7)$ to the circle

$$x^2 + y^2 - 6x + 4y = 12.$$

9. Prove that the circles

$$x^2 + y^2 + 16x + 12y = 0,$$
$$x^2 + y^2 - 8x + 2y + 8 = 0,$$

are tangent to each other. Find the coördinates of the point of contact.

10. Find the shortest distance from the circle

$$x^2 + y^2 + 8x - 6y = 0$$

to the circle

$$x^2 + y^2 - 16x + 2y + 40 = 0.$$

39. Circles determined by three conditions. Every circle has an equation of the form

$$x^2 + y^2 + Dx + Ey + F = 0, \tag{1}$$

where D, E, and F are constants. Three conditions in general determine these three constants, as is illustrated in the following examples.

Example 1. — Find an equation of the circle which passes through the three points $(6, 2)$, $(7, 1)$, $(8, -2)$.

Solution. — The circle has an equation of the form (1). The coördinates of each of the three points must satisfy the equation; hence

$$36 + 4 + 6D + 2E + F = 0,$$
$$49 + 1 + 7D + E + F = 0,$$
$$64 + 4 + 8D - 2E + F = 0.$$

Solving for D, E, F, we find

$$D = -6, \quad E = 4, \quad F = -12$$

Hence the required equation is

$$x^2 + y^2 - 6x + 4y - 12 = 0.$$

Example 2. — Find an equation of the circle whose center lies on the line $2x - y = 3$ and which passes through the points $(-1, -2)$ and $(0, -3)$.

Solution. — The equation has the form (1). The center lies at $(-\frac{1}{2}D, -\frac{1}{2}E)$, and its coördinates must satisfy the equation of the given line; hence

$$-D + \tfrac{1}{2}E = 3.$$

The coördinates of the two given points must satisfy (1); hence

$$1 + 4 - D - 2E + F = 0,$$
$$0 + 9 + 0 - 3E + F = 0.$$

Solving the last three equations, we find

$$D = -2, \quad E = 2, \quad F = -3.$$

The required equation is

$$x^2 + y^2 - 2x + 2y - 3 = 0.$$

Example 3. — Find an equation of the circle inscribed in the triangle whose sides have the equations

$$x + 2y = 5, \quad 2x - y = 5, \quad 2x + y = -5.$$

Solution. — The center lies on the bisectors of the angles of the triangle; two of these bisectors are found to be

$$x - 3y = 0, \quad x = 0.$$

Solving these equations, we find that the center is the point $(0, 0)$. The radius is the distance from a side of the triangle to the center; this distance is found to be $\sqrt{5}$. Hence the required equation of the circle is

$$x^2 + y^2 = 5.$$

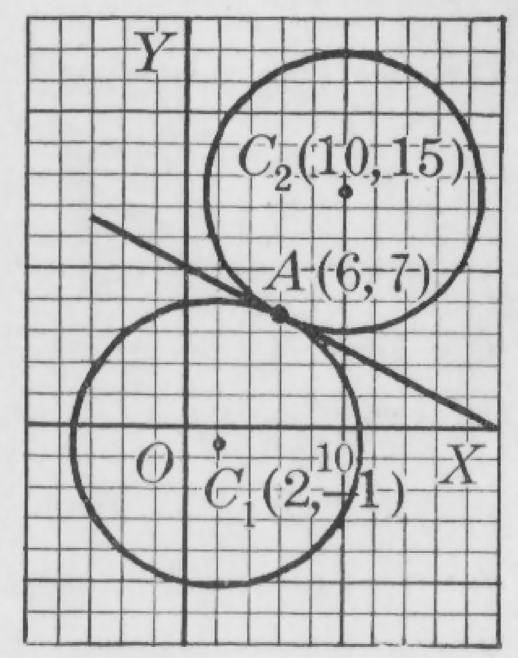
Example 4. — Find an equation of a circle whose radius is $4\sqrt{5}$ and which is tangent to the line $x + 2y = 20$ at the point $A(6, 7)$.

Solution. — We have given $R = 4\sqrt{5}$. Let us find h and k, the coördinates of the center. The distance from the point $C(h, k)$ to the point $A(6, 7)$ is $4\sqrt{5}$; hence

$$(h - 6)^2 + (k - 7)^2 = (4\sqrt{5})^2 = 80. \tag{1}$$

The radius AC is perpendicular to the tangent line $x + 2y = 20$. Hence the slope of the one is the negative reciprocal of the slope of the other, and we have

$$(2) \qquad \frac{k-7}{h-6} = 2.$$



We solve (1) and (2) for h and k. From (2) we have

$$k = 2h - 5.$$

Substituting in (1) and simplifying, we obtain

$$h^2 - 12h + 20 = 0.$$

Hence $h = 2$ or $h = 10$. Corresponding values of k are $k = -1$ or $k = 15$. There are two possible centers, $C_1(2, -1)$ and $C_2(10, 15)$. The equations of the two corresponding circles are

$$(x-2)^2 + (y+1)^2 = 80, \qquad (x-10)^2 + (y-15)^2 = 80.$$

EXERCISES

Find an equation of the circle which passes through the three points given in each of the following Exercises 1–4.

1. $A(0, 0)$, $B(6, 0)$, $C(0, -10)$.

2. $A(3, 0)$, $B(0, 4)$, $C(3, 4)$.

3. $A(2, 2)$, $B(2, 6)$, $C(18, 4)$.

4. $A(-3, 6)$, $B(-2, -6)$, $C(2, 4)$.

Find an equation of the circle which circumscribes the triangle whose vertices are given in each of the following Exercises 5–8.

5. $A(0, 0)$, $B(-8, 0)$, $C(0, 6)$.

6. $A(-6, 0)$, $B(2, 4)$, $C(0, -6)$.

7. $A(-1, -1)$, $B(-1, -3)$, $C(-9, -2)$.

8. $A(3, -6)$, $B(2, 6)$, $C(-2, -4)$.

Find an equation of the circle inscribed in the triangle formed by the lines given in each of the following Exercises 9–12.

9. $x = 0$, $\quad 2x + y = 0$, $\quad 2x - y - 8 = 0$.

10. $x - 2y = 5$, $\quad 2x + y = 5$, $\quad 2x - y + 5 = 0$.

11. $3x + y + 1 = 0$, $\quad x - 3y + 3 = 0$, $\quad x + 3y - 11 = 0$.

12. $3x - 4y - 8 = 0$, $\quad 4x + 3y - 12 = 0$, $\quad 4x - 3y + 36 = 0$.

Find an equation of the circle inscribed in the triangle whose vertices are the points given in each of the following Exercises 13–15.

13. $A(0, 0)$, $B(0, 6)$, $C(8, 0)$.

14. $A(6, 4)$, $B(6, 10)$, $C(14, 4)$.

15. $A(2, 2)$, $B(2, 4)$, $C(10, 2)$

16. Find an equation of the circle whose center is $(2, 5)$ and which is tangent to the line $5x - 12y = 13$.

17. Find an equation of the circle whose center is $(-3, -4)$ and which is tangent to the line $3x + 4y = 20$.

18. Find an equation of each circle which passes through the points $(0, 2)$ and $(2, 4)$ and is tangent to the x-axis.

19. Find an equation of each circle which passes through the points $(-3, 3)$ and $(1, 5)$ and is tangent to the line $4x + 3y + 13 = 0$.

20. Find an equation of the circle which passes through the points $(2, 1)$ and $(4, 7)$ and is tangent to the line $4x - 3y + 5 = 0$.

21. Find an equation of the circle which passes through the points $(2, 3)$ and $(-1, 1)$ and has its center on the line $x - 3y = 11$.

22. Find an equation of the circle which passes through the points $(1, 1)$ and $(3, -2)$ and has its center on the line $3x + y + 11 = 0$.

23. A point moves so that the sum of the squares of its distances from $(2, 4)$ and $(-2, -4)$ is 72. Show that the locus is a circle, and find its center and radius.

24. A point moves so that it is always twice as far from the point $(6, 2)$ as from the point $(2, -2)$. Show that the locus is a circle and find its center and radius.

25. A point moves so that the sum of the squares of its distances from the lines $3x + 4y = 5$ and $4x - 3y = 10$ is always 25. Show that the locus is a circle and find its center and radius.

26. Find an equation of the circle which is tangent to the lines $2x + 3y + 6 = 0$ and $3x + 2y - 9 = 0$ and passes through the point $(-6, -6)$.

27. A point moves so that the square of its distance from the point $(3, 2)$ is proportional to its distance from the line $3x - 4y = 10$. Prove that the locus is a circle.

★ 40. The system $S + kS' = 0$. Let us write the equation of a circle in the abbreviated form

$$(1) \qquad S = 0$$

where S stands for the expression $x^2 + y^2 + Dx + Ey + F$. Similarly we write the equation of another circle

$$(2) \qquad S' = 0$$

where S' stands for $x^2 + y^2 + D'x + E'y + F'$. Consider now the equation

$$(3) \qquad S + kS' = 0.$$

If this is written out at length and terms are rearranged, we have

$$(4) \qquad (1 + k)x^2 + (1 + k)y^2 + (D + kD')x + (E + kE')y + (F + kF') = 0.$$

The question arises: What is the geometric relationship between the locus of equation (3) and the circles (1) and (2)?

If the circles (1) and (2) intersect, the coördinates of each point of intersection must reduce both S and S' to zero; hence $S + kS'$ is also reduced to zero. It follows that *the locus of equation* (3) *passes through the intersections of* (1) *and* (2). This is true regardless of the value of the constant k.

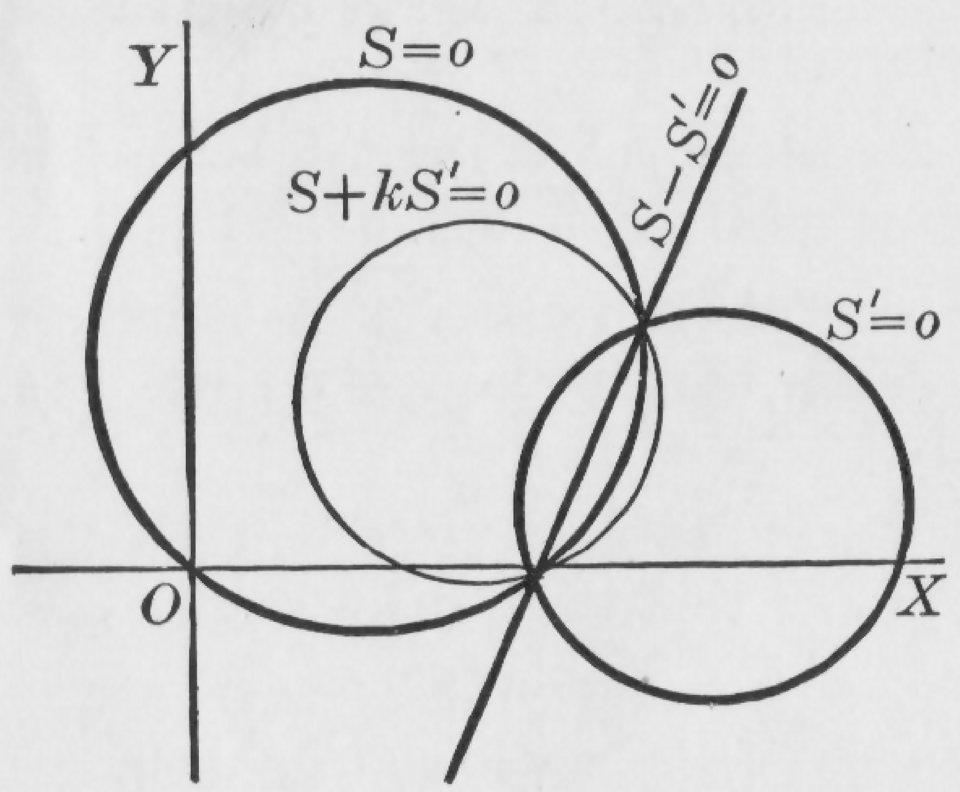


FIG. 46

If $k = -1$ we observe that equation (4) is linear, provided $D + kD'$ and $E + kE'$ are not both zero. Hence in this case equation (3) is the equation of a straight line. If the circles (1) and (2) intersect in two points, this line passes through them and therefore contains the *common chord of the circles* (Fig. 46). Whether the circles intersect or not, the line is called their **radical axis** (see Ex. 8, p. 108).

If $k \neq -1$, we divide equation (4) by $(1 + k)$ and obtain

$$x^2 + y^2 + \text{terms of lower degree} = 0,$$

which is of the form (1) discussed in § 38, page 96. Hence equation (3) may represent (*a*) a circle, (*b*) a point circle, or (*c*) it may have no locus.

If S = 0 *and* S′ = 0 *are equations of intersecting circles then* S + kS′ = 0 *is a circle which passes through their points of intersection if* k *is any constant other than* − 1.

By giving different values to k in equation (3) we get different circles. Thus equation (3) represents a *system of circles,* with k as parameter. Each point of the plane lies on one circle (or on the radical axis) of the system, except for points on the circle (2). To prove this, let $P_1(x_1, y_1)$ be any point in the plane. Substitute $x = x_1, y = y_1$ in (3), designate the resulting equation as $S_1 + kS_1' = 0$, and solve for k; we obtain $k = -S_1/S_1'$. This is possible if P_1 is not on the circle $S' = 0$. For this value of k the circle (3) passes through P_1. If it should happen that $k = -1$, the locus is the radical axis instead of a circle; it is sometimes called a *circle of infinite radius.*

Example. — Given two circles $S = 0$, $S' = 0$ where

$$S \equiv x^2 + y^2 + 2x + 2y - 14,$$
$$S' \equiv x^2 + y^2 - 4x - 4y - 41,$$

find the radical axis, and the circle which passes through the intersections of the circles and the point $(1, -1)$.

Solution. — The radical axis $S - S' = 0$ is

$$6x + 6y + 27 = 0.$$

To determine the circle of the system $S + kS' = 0$ which passes through the point $(1, -1)$, we substitute $x = 1$, $y = -1$ in $S + kS' = 0$. For these values of x and y we find $S = -12$, $S' = -39$; hence

$$-12 - 39k = 0, \quad \text{or} \quad k = -\frac{4}{13}.$$

The required circle is $S - \frac{4}{13}S' = 0$; this reduces to

$$9x^2 + 9y^2 + 42x + 42y - 18 = 0,$$

or

$$\left(x + \frac{7}{3}\right)^2 + \left(y + \frac{7}{3}\right)^2 = \frac{116}{9}.$$

EXERCISES

1. Draw the circles $S + kS' = 0$, where

$$S = x^2 + y^2 - 8x - 9,$$
$$S' = x^2 + y^2 + 8x - 9,$$

when $k = 0, 1, 3, -1, -10$.

2. Draw the circles $S + kS' = 0$, where

$$S = x^2 + y^2 - 6y - 16,$$
$$S' = x^2 + y^2 + 8y,$$

when $k = 0, 3, 8, -1, -3$.

3. Find an equation of the circle which passes through the point (3, 2) and through the intersections of the circles

$$x^2 + y^2 - 4 = 0, \quad x^2 + y^2 + 4x = 0.$$

4. Find an equation of the circle which passes through the point (2, − 2) and through the intersections of the circles

$$x^2 + y^2 - 4x = 0, \quad x^2 + y^2 + 4x = 5.$$

Find the equation of the radical axis and draw the figures for each pair of circles in the following Exercises 5 *and* 6.

5. (*a*) $x^2 + y^2 = 4, \quad x^2 + y^2 - 4x = 5;$
(*b*) $x^2 + y^2 = 4, \quad x^2 + y^2 - 8x + 12y = 10;$
(*c*) $x^2 + y^2 = 4, \quad x^2 + y^2 - 8x + 15y = 0.$

6. (*a*) $x^2 + y^2 = 9, \quad x^2 + y^2 + 6x - 8y = 0;$
(*b*) $x^2 + y^2 = 9, \quad x^2 + y^2 + 6x - 8y + 21 = 0;$
(*c*) $x^2 + y^2 = 9, \quad x^2 + y^2 + 6x - 8y + 24 = 0.$

7. Let $P_1(x_1, y_1)$ be outside the circle $(x - h)^2 + (y - k)^2 - R^2 = 0$. Let t be the length of the tangent from P_1 to the circle (Fig. 47). Prove that $t^2 = (x_1 - h)^2 + (y_1 - k)^2 - R^2$.

8. Let P_1 be a point on the radical axis of the circles $S = 0$, $S' = 0$ and outside the circles. Let t and t' be the lengths of the tangents from P_1 to the respective circles $S = 0$ and $S' = 0$ (Fig. 47). Use the formula of Exercise 7 to prove that $t = t'$.

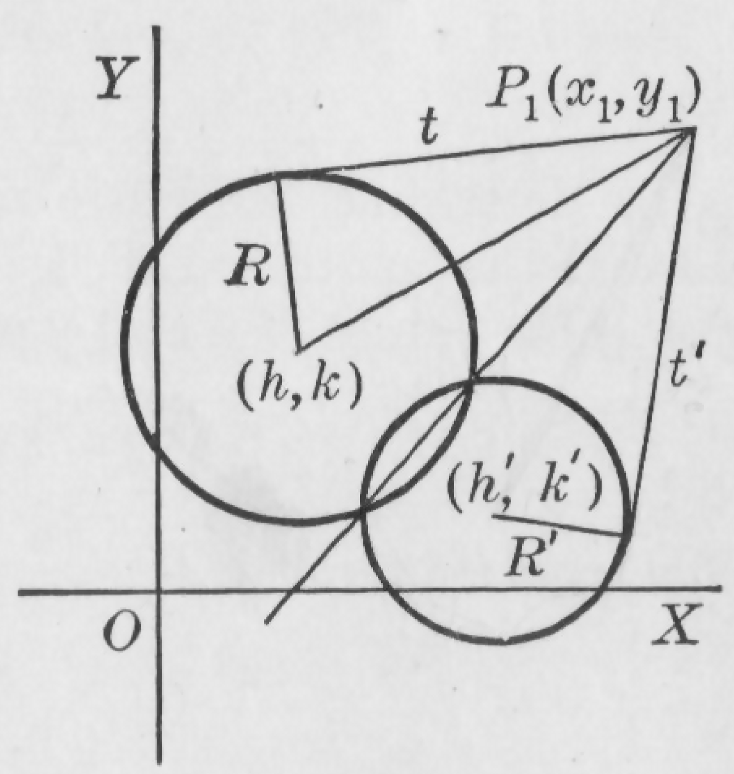


FIG. 47

Use the fact proved in Exercise 7 *to find in each of the following Exercises the point from which the lengths of tangents drawn to the three given circles are equal.*

9. $x^2 + y^2 + 6x + 8y = 0,$
$x^2 + y^2 + 3x + 9y = 1,$
$x^2 + y^2 + 4x + 7y + 9 = 0.$

10. $x^2 + y^2 + 2x - 7y + 4 = 0,$
$x^2 + y^2 + 4x - 6y = 0,$
$x^2 + y^2 - x - 8y + 11 = 0.$

POLAR COÖRDINATES

41. Polar equations of circles. For a given system of polar coördinates let the point $C(a, \alpha)$ be the center and R the radius of a circle. To find an equation of the circle let $P(r, \theta)$ be any point on the circle. Apply the Law of Cosines to the triangle OCP. We obtain the required equation

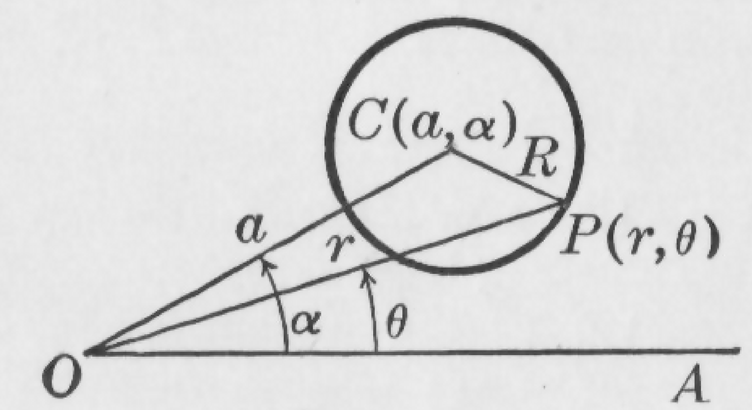


FIG. 48

$$(1) \qquad r^2 - 2ar\cos(\theta - \alpha) + a^2 = R^2.$$

If the center is at the origin, we have $a = 0$, and (1) becomes

$$(2) \qquad r^2 = R^2.$$

If the center is at the point $(R, 0)$ the equation reduces to the simple form

$$(3) \qquad r = 2R\cos\theta.$$

This circle passes through the origin, and its center is on the polar axis.

EXERCISES

1. Find the polar equation of the circle

(*a*) whose center is $\left(2\sqrt{2}, \frac{\pi}{4}\right)$ and whose radius is 2;

(*b*) whose center is $\left(2, \frac{\pi}{3}\right)$ and whose radius is 2;

(*c*) whose center is (2, 0) and whose radius is 2.

2. Find the polar equation of the circle

(a) whose center is $\left(3\sqrt{2}, -\frac{\pi}{4}\right)$ and whose radius is 3;

(b) whose center is $\left(3, -\frac{\pi}{6}\right)$ and whose radius is 3;

(c) whose center is $\left(3, \frac{\pi}{2}\right)$ and whose radius is 3.

3. Show that if the center of a circle of radius R is at the point $\left(R, \frac{\pi}{2}\right)$ it has the polar equation $r = 2R \sin \theta$.

4. Show that if the center of a circle of radius R is at (R, π) the polar equation is $r + 2R \cos \theta = 0$.

Find the center and radius of each of the circles whose polar equations are as follows.

5. $r = 3$. **6.** $r = 6 \cos \theta$.

7. $r = 8 \sin \theta$. **8.** $r = -12 \cos \theta$.

9. $r = -10 \sin \theta$. **10.** $r = 6 \cos \left(\theta - \frac{\pi}{3}\right)$.

11. $r = 8 \cos \left(\theta - \frac{\pi}{4}\right)$. **12.** $r = 10 \cos \left(\theta + \frac{\pi}{3}\right)$.

13. $r^2 - 6r \cos \theta = 16$. **14.** $r^2 - 8r \sin \theta = 9$.

15. $r = 10 \cos \theta - 10\sqrt{3} \sin \theta$. **16.** $r = \cos \theta + \sin \theta$.

17. Derive equation (2), § 41, from equation (3), § 37, by means of the equations, $x = r \cos \theta$, $y = r \sin \theta$.

18. Derive equation (1), § 41, from equation (2), § 37, by means of the equations $x = r \cos \theta$, $y = r \sin \theta$.

19. Show that if in Figure 48 the coördinates of P are $(-r, \pi + \theta)$, equation (1), § 41, follows from the Law of Cosines.

20. Find the distance between the centers of the circles whose polar equations are

$$r = 10 \sin \theta, \qquad r = -12 \cos \theta.$$

21. Proceed as in Exercise 20 when the equations are

$$r^2 - 6r \cos \theta = 16, \qquad r = 8 \cos \left(\theta - \frac{\pi}{4}\right).$$

22. Find a polar equation of the straight line which passes through the centers of the circles

$$r^2 - 8r \sin \theta = 9, \qquad r = 2 \cos \theta + 2 \sin \theta.$$

MISCELLANEOUS EXERCISES

In some of the following Exercises the axes of coördinates are not given. The student should choose them so that the coördinates and equations will be as simple as possible.

1. A point moves so that the sum of the squares of its distances from two given perpendicular lines is a constant. Prove analytically that the locus is a circle.

2. A point moves so that the ratio of its distances from two fixed points is a constant, k. Prove analytically (a) that if $k = 1$ the locus is a straight line, and (b) that if $k \neq 1$ the locus is a circle.

3. A point moves so that the square of its distance from a fixed point is proportional to its distance from a fixed line. Prove analytically that the locus is a circle.

4. A point moves so that the sum of the squares of its distances from two fixed points is constant. Prove analytically that the locus is a circle.

5. The ends of a rod slide along two fixed perpendicular wires. Prove analytically that the mid-point of the rod moves in a circle.

6. Show that the locus of the equation

$$\begin{vmatrix} x^2 + y^2 & x & y & 1 \\ x_1^2 + y_1^2 & x_1 & y_1 & 1 \\ x_2^2 + y_2^2 & x_2 & y_2 & 1 \\ x_3^2 + y_3^2 & x_3 & y_3 & 1 \end{vmatrix} = 0$$

passes through the three points $P_1(x_1, y_1)$, $P_2(x_2, y_2)$, $P_3(x_3, y_3)$ and that it is a circle if the determinant

$$\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$$

has a value different from zero. Give the geometrical significance of the value of the latter determinant and of its vanishing (see p. 79).

7. Let $P_1(x_1, y_1)$ be a point outside the circle

$$x^2 + y^2 + Dx + Ey + F = 0.$$

Prove that if t is the length of the tangent from P_1 to the circle, then

$$t^2 = x_1^2 + y_1^2 + Dx_1 + Ey_1 + F.$$

8. Prove that the locus of a point which moves so that the lengths of the tangents drawn from it to two given circles are equal is (*a*) the radical axis of the circles if they do not intersect, or (*b*) the part of the radical axis exterior to the circles if they intersect.

Hint. Use results of Exercise 7.

9. Prove that if two circles are concentric they have no radical axis.

10. Show analytically that the radical axis of two circles which are not concentric cuts their line of centers at right angles.

11. If three circles are such that each pair has a radical axis, prove that the three radical axes thereby determined intersect in a point or are parallel.

12. Find the points of intersection of the circles

$$x^2 + y^2 + 4x = 21,$$
$$x^2 + y^2 - 8y = 9.$$

Did you use the equation of the radical axis of the circles at any step in your solution?

13. Two circles intersect **orthogonally** if the angle between their tangents at a point of intersection is a right angle. Prove that the circles

$$x^2 + y^2 + Dx + Ey + F = 0,$$
$$x^2 + y^2 + D'x + E'y + F' = 0$$

intersect orthogonally if and only if

$$DD' + EE' = 2(F + F').$$

14. Prove that the circles

$$x^2 + y^2 = 2kx,$$
$$x^2 + y^2 = 2k'y,$$

where k and k' are constants, intersect orthogonally (see Ex. 13). Draw the circles for $k = 2, 4, 6$ and $k' = 2, 4, 6$, as a check.

15. Find an equation of a circle which passes through the point (0, 0) and cuts orthogonally both of the circles of Exercise 12. Is there a circle which passes through the point (3, 0) and cuts orthogonally both of the circles of Exercise 12? Draw a figure.

16. Find an equation of the circle which passes through the intersections of the circles of Exercise 12 and cuts the first one orthogonally. Also one through the intersections which cuts the second circle orthogonally. Draw a figure.

17. Prove that every circle which passes through the points (3, 0) and (12, 0) cuts the circle $x^2 + y^2 = 36$ orthogonally (see Ex. 13).

18. Prove that every circle which passes through the points $(ka, 0)$ and $(a/k, 0)$ cuts the circle $x^2 + y^2 = a^2$ orthogonally.

19. Find an equation of the circle K which passes through the mid-points D, E, F of the sides of the triangle whose vertices are $A(-12, 0)$, $B(6, 0)$, $C(0, 6)$. Drop perpendiculars l_1, l_2, l_3 from A, B, C on the opposite sides; let the feet be G, H, I. Prove that the circle K passes through G, H, and I. Let Q be the point of intersection of l_1, l_2, l_3, and let L, M, N be the mid-points of the lines joining A, B, C with Q. Prove that L, M, N lie on the circle K. Draw an accurate figure, locating all of the points and draw K.

20. Prove that the circle K which passes through the mid-points D, E, F of the sides of any triangle ABC also passes through the following six points: the feet G, H, I of the perpendiculars from the vertices to the opposite sides, and the mid-points L, M, N of the lines joining the vertices to the point Q of intersection of the perpendiculars from the vertices to the opposite sides of the triangle ABC.

Hint. Choose coördinate axes so that the vertices are $A(a, 0)$, $B(b, 0)$, $C(0, c)$. Find the equation of the circle through D, E, F and prove that I and L lie on it.

Note. This circle K which passes through the nine points D, E, F, G, H, I, L, M, N is called the **nine point circle** of the triangle ABC.

21. Prove that if the equations

$$x^2 + y^2 + Dx + Ey + F = 0,$$
$$x^2 + y^2 + D'x + E'y + F' = 0,$$

have the same locus, then the equations are identical, that is, $D = D'$, $E = E'$, $F = F'$.

22. Prove that if the equations

$$x^2 + y^2 = r,$$
$$ax^2 + bxy + cy^2 + dx + ey = f,$$

are equations of the same circle, then

$$a : 1 = c : 1 = f : r, \text{ and } b = d = e = 0.$$

Hint. The points $(r, 0)$, $(-r, 0)$, $(0, r)$, $(0, -r)$ are on the circle; also if (x_1, y_1) is on the circle, so also is $(x_1, -y_1)$.

CHAPTER VI

STANDARD EQUATIONS OF THE CONIC SECTIONS

42. Introduction. The curves of intersection of a plane and a right circular cone are called **conic sections or conics.** If the plane cuts across one nappe of the cone, the section is an ellipse (curve E, Fig. 49). If the plane is parallel to a line in the surface of the cone, the section is a parabola (curve P, Fig. 49). If the plane cuts both nappes of the cone the section is a hyperbola (curve H, Fig. 49).

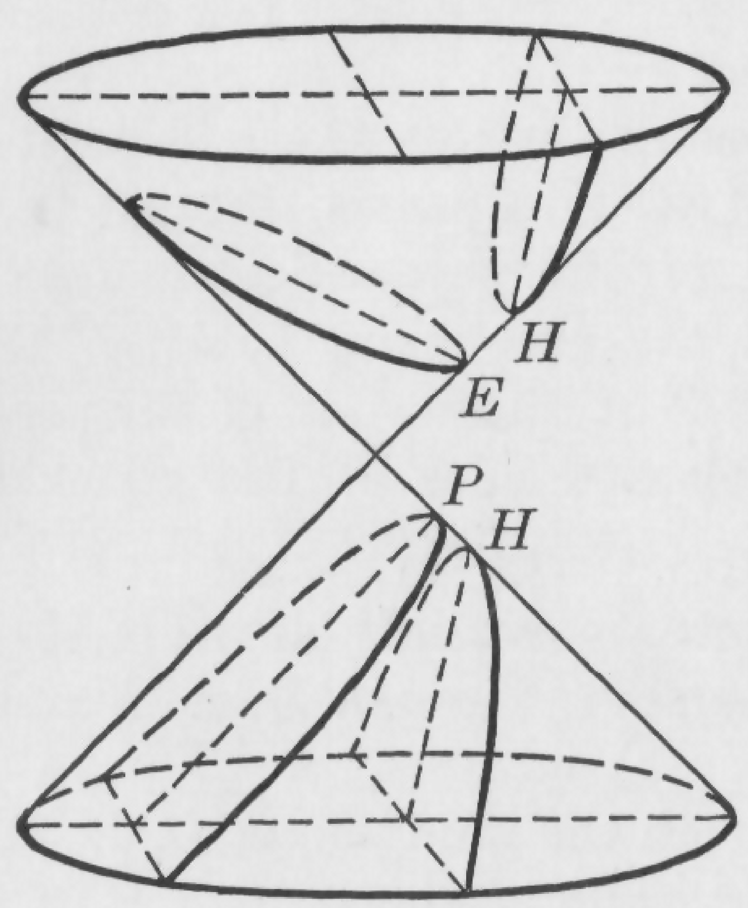


FIG. 49

The preceding definitions explain the origin of the name "conic" as applied to these curves. While it is possible to proceed from these definitions to a study of the curves, it turns out to be simpler, and it is customary, to start from other definitions which will be given in following sections. In Chapter XVII, § 154, we shall show that the two sets of definitions are equivalent.

These curves were extensively studied by the ancient Greeks, and are of great interest in mathematics, both pure and applied, because of their important properties. They are in many respects the simplest types of curves after the straight line and the circle. This simplicity will present itself to us here through the fact that they all have equations of the second degree, coming under the general form

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0.$$

It will be shown, moreover, in Chapter XII, §§ 109, 110, that the locus of an equation of this type is always a conic (under special conditions it degenerates into one or two lines, or a point, or the equation has no locus).

THE PARABOLA

43. Standard equation of a parabola. The usual definition of a parabola is as follows: *It is the locus of a point which moves in a plane so that its distance from a fixed point constantly equals its distance from a fixed line.* The fixed point is called the **focus** and the fixed line the **directrix** of the parabola.

Let F be the focus and MN the directrix of a parabola (Fig. 50). The points of the parabola are equidistant from F and MN. A number of such points are shown in the figure. One of them is at the mid-point, V, of the perpendicular FD from F to MN. It is called the **vertex** of the parabola. The line VF produced indefinitely through F is called the **axis** of the parabola. The distance DF will be denoted by p.

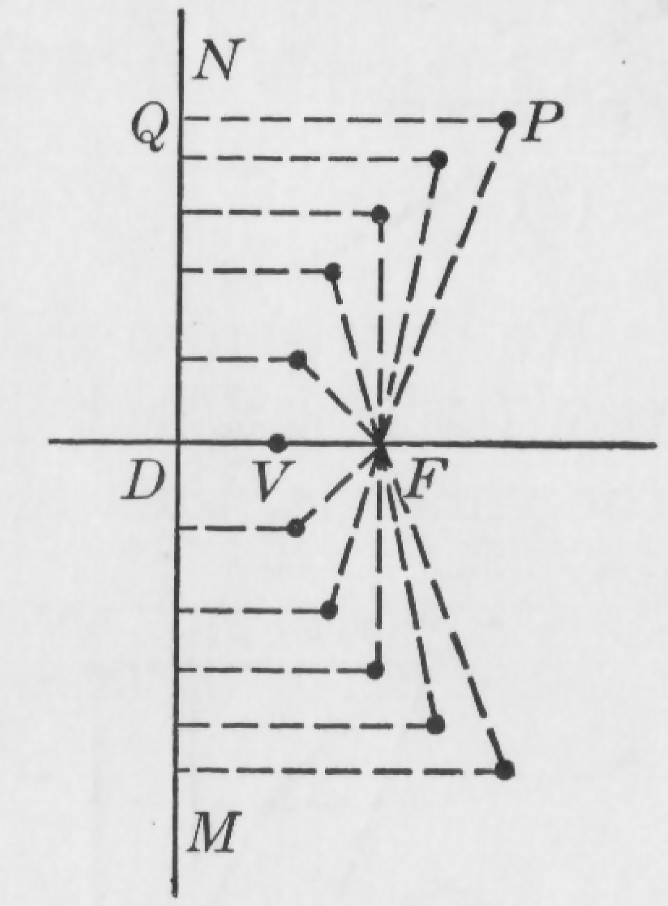


FIG. 50

Methods of constructing a parabola are given in § 58, pages 135, 136, to which the student may now refer.

The equation of the parabola turns out to be especially simple if we take V as the origin of a system of rectangular coördinates, and the line through V and F as a coördinate axis.

If the focus F is on the positive x-axis, the coördinates of F are $(p/2, 0)$. Let $P(x, y)$ be any point on the parabola. Let PQ be the perpendicular from P to the directrix. Then the distances FP and QP are equal; hence

$$(1) \qquad FP^2 = QP^2.$$

We have

$$FP^2 = \left(x - \frac{p}{2}\right)^2 + y^2.$$

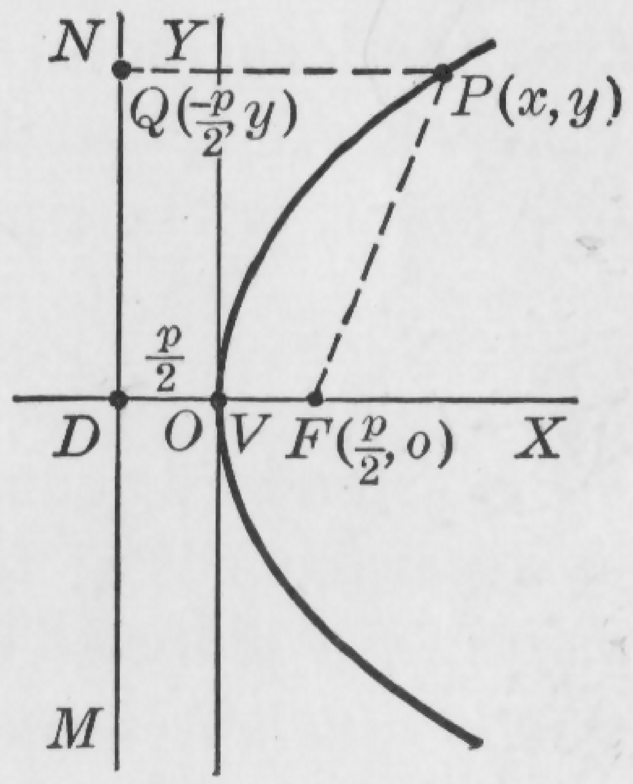


Fig. 51

It is seen that the coördinates of Q are $(-p/2, y)$ and hence

$$QP^2 = \left(x + \frac{p}{2}\right)^2 + (y - y)^2.$$

On equating these expressions and expanding we obtain

$$x^2 - px + \frac{p^2}{4} + y^2 = x^2 + px + \frac{p^2}{4},$$

and hence

(2) $$\boldsymbol{y^2 = 2px.}$$

We have shown that if $P(x, y)$ is on the parabola, equation (2) is satisfied. If we take any point $P(x, y)$ not on the parabola, then equation (1) is not satisfied and consequently equation (2) does not hold. Thus equation (2) is satisfied by points on the parabola and by no other points.

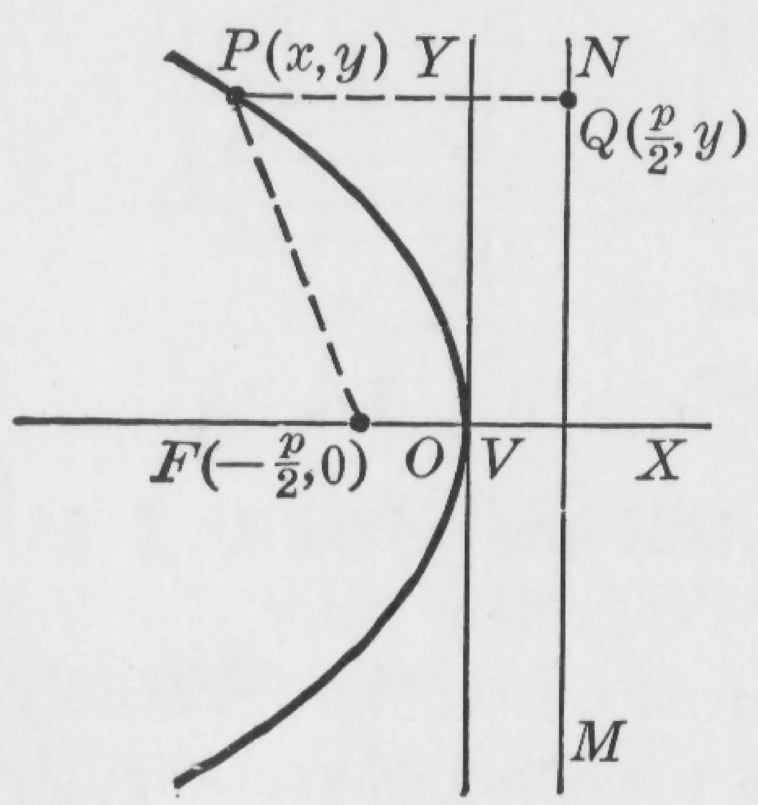


Fig. 52

Equation (2) is called the **standard equation** of a parabola. The equation of the directrix is seen to be

$$x = -\frac{p}{2}.$$

The focus is at the point $(p/2, 0)$, the vertex is at $(0, 0)$.

If the vertex is at the origin and the focus is on the negative x-axis, as illustrated in Figure 52, the coördinates of F are $(-p/2, 0)$. Then if $P(x, y)$ is on the locus, $FP^2 = QP^2$. We have

$$FP^2 = \left(x + \frac{p}{2}\right)^2 + y^2.$$

The coördinates of Q in the figure are $(p/2, y)$; hence

$$QP^2 = \left(x - \frac{p}{2}\right)^2 + (y - y)^2.$$

Since $FP^2 = QP^2$, we have

$$x^2 + px + \frac{p^2}{4} + y^2 = x^2 - px + \frac{p^2}{4},$$

and hence

(3) $$\boldsymbol{y^2 = -2px.}$$

The student should show that if the vertex is at the origin and if the focus is on the positive y-axis, the corresponding equation of the parabola is

(4) $$\boldsymbol{x^2 = 2py;}$$

and that if the focus is on the negative y-axis the equation is

(5) $$\boldsymbol{x^2 = -2py.}$$

44. Discussion of the parabola. Consider the standard equation of a parabola,

$$y^2 = 2px.$$

Since p is positive and y^2 positive or zero, we observe that x must be positive or zero. Hence the curve does not extend to the left of the origin, that is, of the vertex.

When we substitute in the equation any positive value x_1 for x, and solve for y, we obtain two values, equal except for sign, $y = \pm y_1$, where $y_1 = \sqrt{2px_1}$. The points (x_1, y_1) and $(x_1, -y_1)$ both lie on the curve (Fig. 53). Hence the curve is symmetrical with respect to the x-axis, on which lies the axis of the parabola.

If we substitute a succession of larger and larger values of x in the equation, and solve for y, we get larger and larger values, increasing indefinitely with x. The curve extends indefinitely far, receding indefinitely from the axis.

The line segment through the focus perpendicular to the axis and terminating on the parabola, LR in Figure 53, is called the **latus rectum** of the parabola. The x-coördinate of an end of the latus rectum is $p/2$. Substituting in the equation and solving, we find $y = \pm p$. Hence the *length of the latus rectum is* $2p$, the coefficient of x in the standard equation.

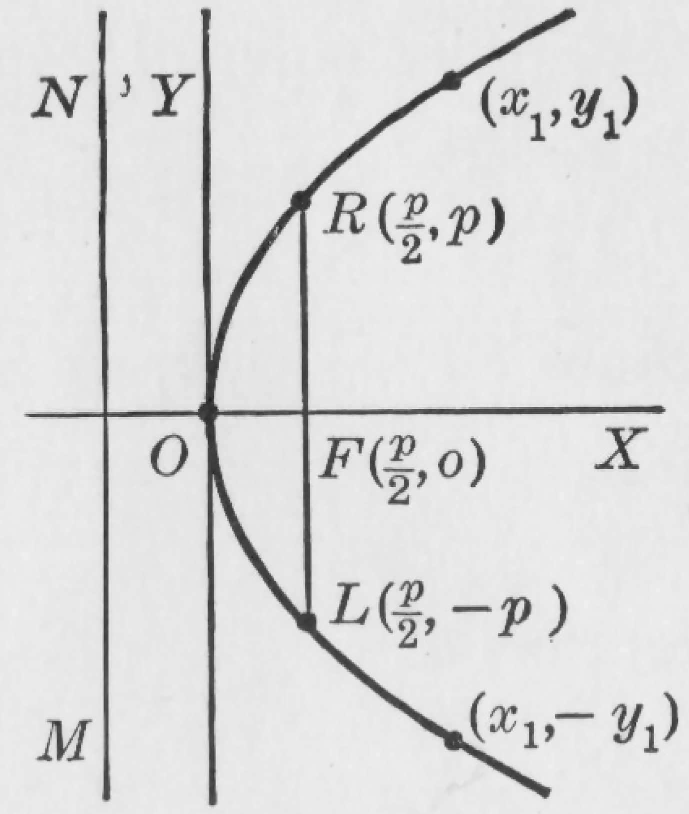


FIG. 53

EXERCISES

1. Derive the equation of the parabola whose vertex is at the origin and focus at $(0, p/2)$.

2. Derive the equation of the parabola whose vertex is at the origin and focus at $(0, -p/2)$.

Draw the locus of the equations in the following Exercises 3–6. *Locate the vertex and focus, and draw the directrix and latus rectum in each case.*

3. (a) $y^2 = 8x$;
(b) $y^2 + 8x = 0$;
(c) $x^2 - 4y = 0$.

4. (a) $x^2 + 12y = 0$;
(b) $2y^2 + x = 0$;
(c) $2x^2 + y = 0$.

5. (a) $y^2 = 10x$;
(b) $x^2 + 4y = 0$;
(c) $y^2 = -10x$.

6. (a) $x^2 = 10y$;
(b) $x^2 = -10y$;
(c) $4y^2 + x = 0$.

7. Find the equation of each of the parabolas described as follows:
(a) vertex $(0, 0)$, focus $(5, 0)$;
(b) focus $(4, 0)$, directrix, $x + 4 = 0$;
(c) vertex $(0, 0)$, directrix, $y - 4 = 0$;
(d) focus $(0, 10)$, directrix, $y + 10 = 0$.

8. Find the equation of each of the parabolas described as follows:
(a) vertex at the origin, focus at $(6, 0)$;
(b) focus $(-4, 0)$, directrix, $x = 4$;
(c) focus $(0, 6)$, directrix, $y = -6$;
(d) focus $(0, -4)$, latus rectum $= 16$, directrix above and parallel to the x-axis.

9. From the definition of a parabola derive the equation of each of the parabolas described as follows:
(a) focus $(4, 4)$, directrix, $x = 0$;
(b) vertex $(2, 4)$, directrix, $x = -2$.

10. Proceed as in Exercise 9 with the parabolas described as follows:
(a) focus $(0, 4)$, vertex $(0, 8)$;
(b) vertex $(-4, -2)$, directrix, $y = 6$.

11. Find the equation of the parabola whose focus is $(0, 0)$ and whose directrix is $3x + 4y = 12$.

12. Find an equation of the locus of a point which moves so that its distance from the point $(0, 8)$ is four units greater than its distance from the line $y + 4 = 0$.

13. Find an equation of the locus of a point which moves so that its distance from the point $(-3, 0)$ is two units less than its distance from the line $x - 5 = 0$.

45. Properties and applications. The parabola and other conics were studied by the ancient Greeks by the methods of elementary geometry, and many properties of these curves were thus discovered. We shall state here two important properties of the parabola to be proved later.

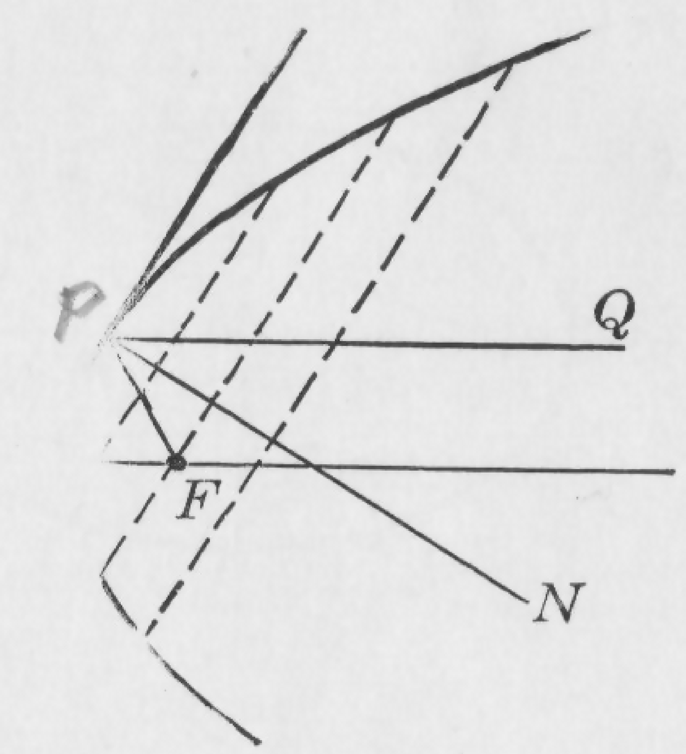


FIG. 54

1. The locus of the mid-points of parallel chords is a straight line parallel to the axis of the parabola (Chapter XI, § 99).

2. At any point P of the parabola (Fig. 54) draw the focal radius FP and a line PQ parallel to the axis of the parabola. Then the angle FPQ is bisected by the line PN (the normal at P) which is perpendicular to the tangent at P (Chapter X, § 95).

The parabola is a curve frequently encountered in applied mathematics. For example, the cable of a suspension bridge whose weight is uniformly distributed over the length of the bridge takes the form of a parabola. The path of a

projectile, such as a cannon ball or a baseball, is an arc of a parabola (neglecting resistance of the air and minor forces).

If a parabola is revolved about its axis a parabolic surface is generated. Rays of light emanating from the focus of a parabolic mirror are reflected in rays parallel to the axis, and rays parallel to the axis on striking the mirror are reflected to the focus. These facts are consequences of the second geometric property stated above. On account of these properties parabolic mirrors are used in searchlights and astronomical telescopes.

If a pan of water is rotated about a vertical axis the surface of the water assumes a parabolic shape.

THE ELLIPSE

46. Standard equation of an ellipse. *If a point moves in a plane so that the sum of its distances from two fixed points is constant, the locus is an* **ellipse.** The two fixed points are called the **foci** of the ellipse. The distances from the foci to a point are called the **focal radii** of the point; in Figure 55, they are $\overline{F'P}$ and $\overline{FP}$, where F' and F are the foci and P is the point. We shall denote the sum of the focal radii of a point on the ellipse by $2a$, and the distance between the foci by $2c$. It is obvious that $2c$ is less than or equal to $2a$, and hence $c \leqq a$.

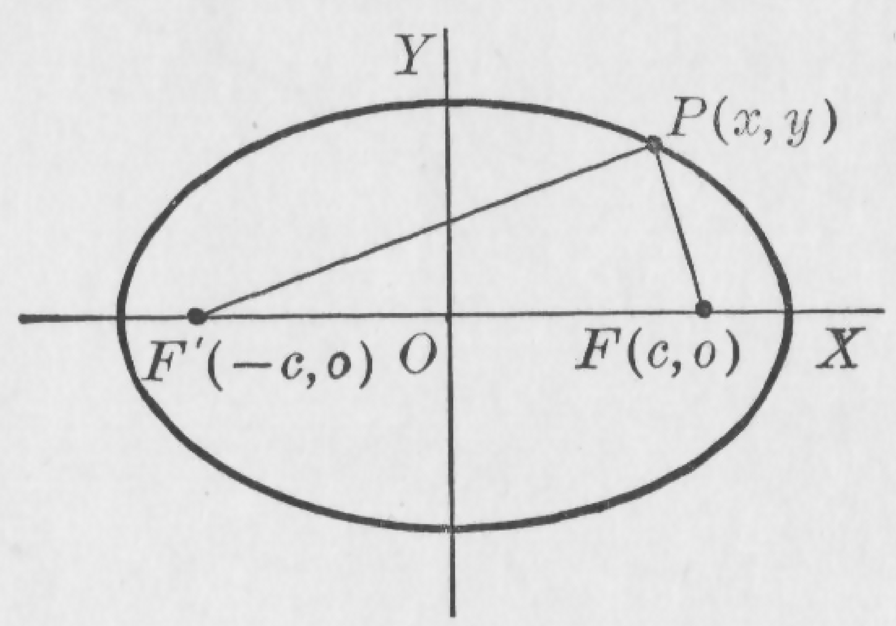


Fig. 55

To obtain a simple equation for an ellipse, let the mid-point of the line joining the foci F' and F be taken as origin, and the line through F' and F as the x-axis. The coördinates of the foci are $(\pm\, c, 0)$. Let $P(x, y)$ be any point on the ellipse. Then, by definition,

(1) $$\overline{F'P} + \overline{FP} = 2a,$$

or $$\sqrt{(x + c)^2 + y^2} + \sqrt{(x - c)^2 + y^2} = 2a.$$

Hence

(2) $$\sqrt{(x + c)^2 + y^2} = 2a - \sqrt{(x - c)^2 + y^2}.$$

Squaring and simplifying, we obtain

$$4cx = 4a^2 - 4a\sqrt{(x - c)^2 + y^2},$$

(3) $$a\sqrt{(x - c)^2 + y^2} = a^2 - cx.$$

If we square again and collect terms, we have

(4) $$(a^2 - c^2)x^2 + a^2y^2 = a^4 - a^2c^2.$$

Division by the right-hand member gives us

$$\frac{x^2}{a^2} + \frac{y^2}{a^2 - c^2} = 1.$$

When $c < a$, $a^2 - c^2$ is positive, and we may write for simplicity

(5) $$b^2 = a^2 - c^2;$$

then the equation of the ellipse becomes

(6) $$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$$

This is called the **standard equation** of the ellipse.

It may be shown that points not on the ellipse do not satisfy this equation (see Exercises 33, 34, p. 122).

If the foci are on the y-axis at $(0, c)$ and $(0, -\,c)$ the equation of the ellipse is

(7) $$\frac{x^2}{b^2} + \frac{y^2}{a^2} = 1, \quad b^2 = a^2 - c^2.$$ $a^2 > b^2$

47. Discussion of the ellipse. It is fairly obvious from the definition of the ellipse that the curve is symmetric with respect to the line passing through the foci F' and F, to the perpendicular bisector of $F'F$, and to the mid-point of $F'F$.

These facts concerning symmetry will now be proved analytically.

First let us recall definitions of symmetry. Two points P and P_1 are symmetric with respect to a line if the line is the perpendicular bisector of PP_1; they are symmetric with respect to a point O if O is the mid-point of PP_1. Thus, in Figure 56, P and P_1 (also P_2 and P_3) are symmetric with respect to the line $F'F$, and P and P_3 (also P_2 and P_1) are symmetric with respect to O.

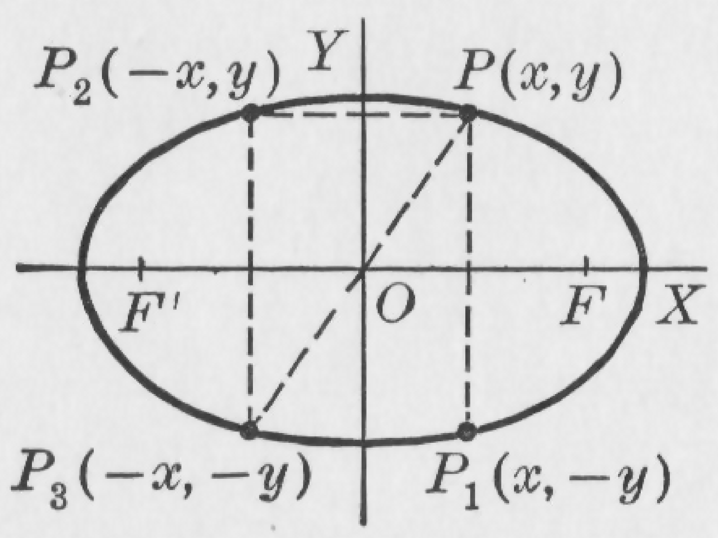


FIG. 56

A curve is symmetric with respect to a line (or point) if the points of the curve can be associated in pairs such that the points of each pair are symmetric with respect to the line (or point); otherwise stated, if P is any point on a curve, this curve being symmetric with respect to a line (or point), then the point P_1 which is symmetric to P is also on the curve.

To prove that the ellipse is symmetric with respect to the line through its foci, we choose coördinate axes as in § 46, and observe (Fig. 56) that we must prove that if $P(x, y)$ lies on the ellipse then $P_1(x, -y)$ does also. Thus by hypothesis (x, y) satisfies the equation of the ellipse

$$(1) \qquad \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

and we must prove that $(x, -y)$ also satisfies it, that is, that

$$(2) \qquad \frac{x^2}{a^2} + \frac{(-y)^2}{b^2} = 1.$$

Since the latter equation follows at once from the former, the symmetry exists.

To prove that the curve is symmetric with respect to the perpendicular bisector of $F'F$ and with respect to the mid-point of $F'F$, we must show similarly (see Fig. 56) that if (x, y) satisfies equation (1) then $(-x, y)$ and $(-x, -y)$ do also, that is, that

$$\frac{(-x)^2}{a^2} + \frac{y^2}{b^2} = 1 \quad \text{and} \quad \frac{(-x)^2}{a^2} + \frac{(-y)^2}{b^2} = 1.$$

These relations follow at once from (1) and hence the symmetry exists.

The chord through the two foci of the ellipse is called the **major axis** of the ellipse. To find its length we substitute $y = 0$ in equation (1) and obtain $x = \pm a$; hence the length of the major axis $A'A$ (Fig. 57) is $2a$.

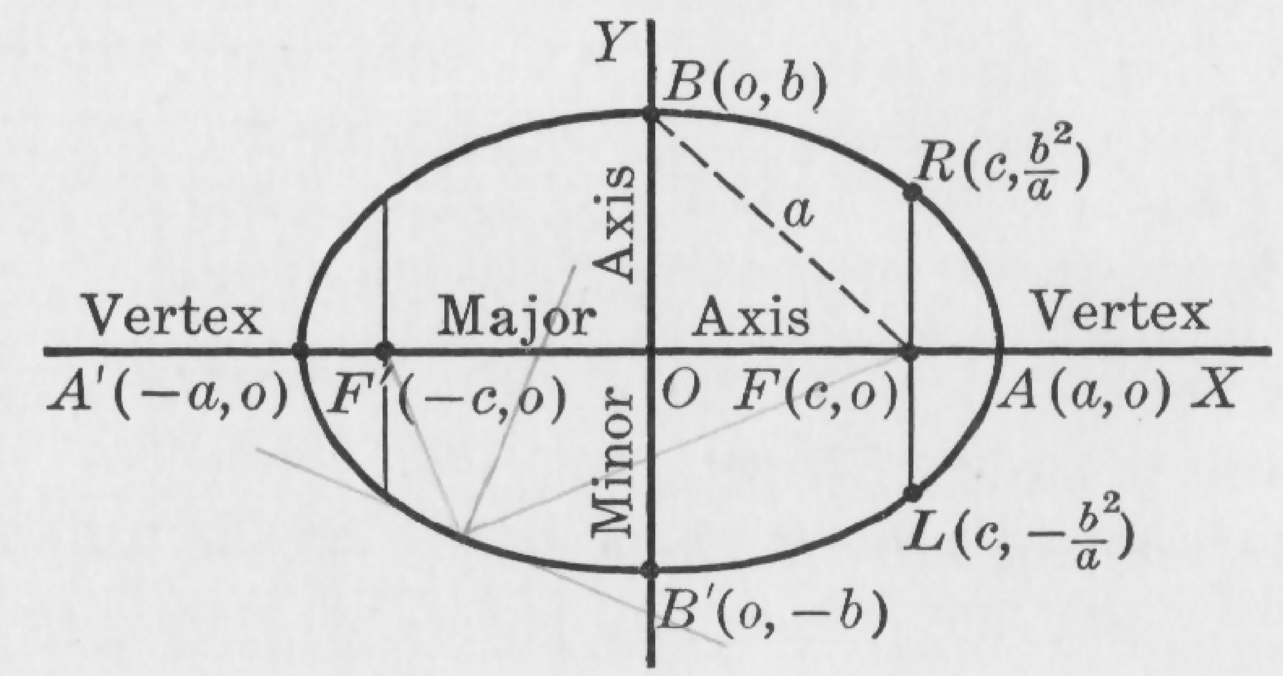


FIG. 57

The chord of the ellipse which is the perpendicular bisector of the major axis is the **minor axis** of the ellipse. To find its length, substitute $x = 0$ in (1) and obtain $y = \pm b$; hence the length of the minor axis $B'B$ is $2b$. Since, by (5), § 46, $b^2 = a^2 - c^2$, it is seen that the minor axis is always shorter than the major axis if c is not zero.

The point of intersection of the two axes of the ellipse is called its **center.** It bisects every chord which passes through it.

An end-point of the major axis is called a **vertex** of the ellipse. The curve bends most sharply at the vertices.

The chord LR (Fig. 57) through a focus perpendicular to the major axis is called a **latus rectum** of the ellipse. To find

its length, we substitute the abscissa of R, which is $x = c$, in the standard equation (1) and solve for y. We have

$$y^2 = b^2\left(1 - \frac{c^2}{a^2}\right) = b^2\left(\frac{a^2 - c^2}{a^2}\right) = \frac{b^4}{a^2};$$

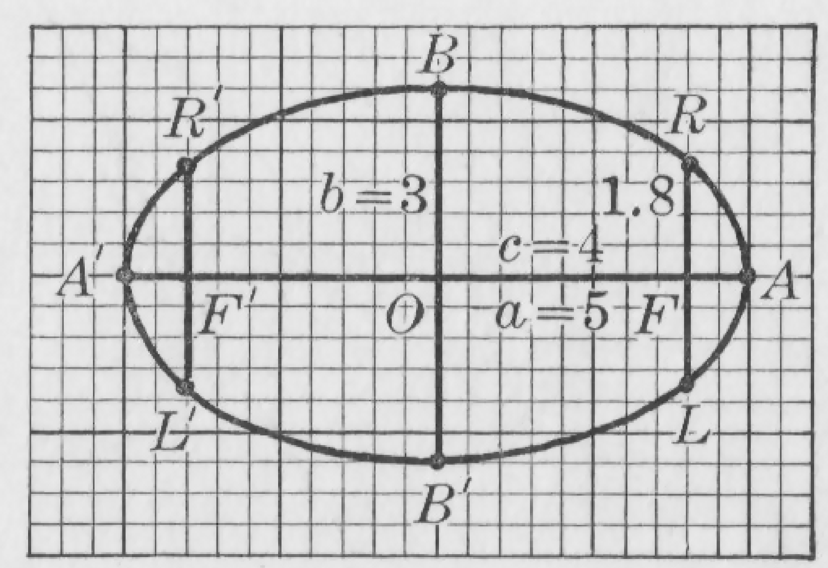
and hence

$$y = \pm \frac{b^2}{a}.$$

The coördinates of R are $(c, b^2/a)$, and thus *the length of the latus rectum is* $2b^2/a$.

Example 1. — Find an equation of the ellipse whose foci are $(\pm 4, 0)$ and whose vertices are $(\pm 5, 0)$. Find the lengths of the axes and the latus rectum; draw the curve.

Solution. — The origin is the center of the ellipse, the x-axis the major axis. We have $a = 5$, $c = 4$. Hence the length of the major axis is $2a = 10$. Since $b^2 = a^2 - c^2$, we have $b = 3$. Thus the length of the minor axis is 6, and the length of the latus rectum is $2b^2/a = 18/5$. The standard equation of the curve becomes



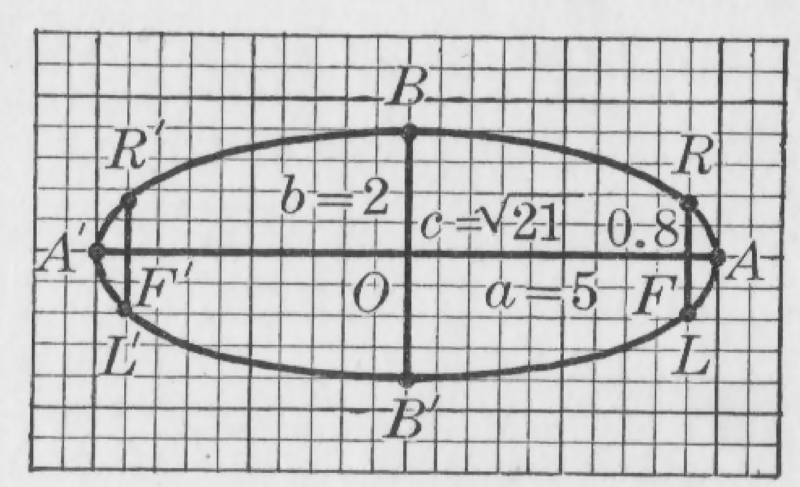
$$\frac{x^2}{25} + \frac{y^2}{9} = 1.$$

To sketch the curve draw the axes $A'A$ and $B'B$, and the latus rectum for each focus, LR and $L'R'$; then draw an oval figure through the end-points A, R, B, R', A', L', B', L.

Example 2. — Find the axes, vertices, and foci of the ellipse having the equation

$$4x^2 + 25y^2 = 100.$$

Solution. — Dividing by 100, we get the standard equation of an ellipse



$$\frac{x^2}{25} + \frac{y^2}{4} = 1.$$

Hence the major axis is on the x-axis, the minor axis on the y-axis, the center at the origin. We have $a^2 = 25$, $b^2 = 4$: thus the vertices are

also 115

the points $(\pm 5, 0)$, the lengths of the axes are 10 and 4, and the length of the latus rectum is $8/5$. Since $c^2 = a^2 - b^2$, we have $c = \sqrt{21}$; thus the foci are the points $(\pm \sqrt{21}, 0)$.

EXERCISES

1. Prove that if the center of an ellipse is at the origin, the foci at $(0, \pm c)$, and the vertices at $(0, \pm a)$, the equation of the ellipse is, as stated on page 117,

$$\frac{x^2}{b^2} + \frac{y^2}{a^2} = 1, \quad \text{where} \quad b^2 = a^2 - c^2.$$

2. Show that the distance from a focus to an end of the minor axis is a.

Find an equation of each of the ellipses satisfying the following conditions. Sketch the curves.

3. The foci are $(\pm 5, 0)$; the vertices are $(\pm 13, 0)$.

4. The foci are $(\pm 12, 0)$; the vertices are $(\pm 13, 0)$.

5. The foci are $(\pm 12, 0)$; the length of the minor axis is 32.

6. The foci are $(\pm 8, 0)$; the length of the minor axis is 12.

7. The foci are $(0, \pm 3)$; the length of the major axis is 10.

8. The foci are $(0, \pm 8)$; the length of the major axis is 34.

9. The vertices are $(0, \pm 17)$; length of latus rectum is $50/17$.

10. The vertices are $(0, \pm 17)$; length of latus rectum is $450/17$.

Sketch the curves for each of the equations in Exercises 11–30.

11. $9x^2 + 16y^2 = 144.$	**12.** $4x^2 + 9y^2 = 36.$
13. $16x^2 + 25y^2 = 400.$	**14.** $25x^2 + 36y^2 = 900.$
15. $9x^2 + 25y^2 = 900.$	**16.** $4x^2 + 25y^2 = 625.$
17. $x^2 + 49y^2 = 196.$	**18.** $x^2 + 64y^2 = 256.$
19. $81x^2 + 100y^2 = 8100.$	**20.** $100x^2 + 121y^2 = 12100.$
21. $4x^2 + y^2 = 144.$	**22.** $25x^2 + 9y^2 = 225.$
23. $36x^2 + 25y^2 = 400.$	**24.** $64x^2 + 25y^2 = 1600.$
25. $49x^2 + 36y^2 = 900.$	**26.** $16x^2 + 9y^2 = 576.$
27. $100x^2 + y^2 = 100.$	**28.** $400x^2 + y^2 = 400.$
29. $441x^2 + 400y^2 = 6400.$	**30.** $400x^2 + 361y^2 = 6400.$

31. Find the locus of a point which moves so that its distance from the point (3, 0) is always one-half of its distance from the line $x = 12$.

32. Find the locus of a point which moves so that its distance from the point (2, 0) is always one-fifth of its distance from the line $x = 50$.

33. Prove that a point $P_1(x_1, y_1)$ which is not on the ellipse defined in § 46 does not satisfy the equation

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1. \tag{6}$$

Hint. If P_1 is not on the ellipse, the sum of the focal radii is $2a'$ where $a' \neq a$. Then (x_1, y_1) satisfies the equation

$$\frac{x^2}{a'^2} + \frac{y^2}{b'^2} = 1, \quad \text{where} \quad b'^2 = a'^2 - c^2.$$

Show that therefore (x_1, y_1) cannot satisfy (6).

34. Solve Exercise 33 by proving that if $P(x, y)$ satisfies equation (6) then

$$\overline{F'P} + \overline{FP} = 2a.$$

Hint. Retrace the steps from (6) to (1) in § 46, giving attention to the $\pm$ signs which enter whenever you extract a square root.

48. Limiting forms of an ellipse. Eccentricity. If $c = 0$, then $a = b$ and the ellipse is a circle. If c is much smaller than a, that is to say, if the ratio c/a is near zero, then b and a are nearly equal, and the ellipse is nearly circular.

If $c = a$, then $b = 0$, and the ellipse is the straight line segment joining the foci. If the ratio c/a is near unity, the ellipse is a very flat oval.

The ratio c/a is called the **eccentricity** of the ellipse, and is denoted by e:

$$e = c/a, \quad \text{or} \quad c = ae.$$

Since c cannot be greater than a, *the eccentricity of an ellipse cannot be greater than unity.*

49. Directrix of an ellipse. An important property of the ellipse is given in the following theorem:

If a point moves so that the ratio of its distance from the point F(ae, 0) *to its distance from the line* x = a/e *is always equal to the constant* e, *less than unity, the locus is the ellipse*

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, \quad \textit{where} \quad b^2 = a^2(1 - e^2). \tag{1}$$

The point F *is a focus of the ellipse, and the constant* e *is the eccentricity.*

The line $x = a/e$ is called the **directrix** of the ellipse corresponding to the focus F.

To prove the theorem, let $P(x, y)$ be any point on the locus. Draw the perpendicular PD to the directrix. From the hypothesis we have

$$FP^2 = e^2 PD^2. \tag{2}$$

We see that

$$FP^2 = (x - ae)^2 + y^2,$$

$$PD^2 = \left(\frac{a}{e} - x\right)^2.$$

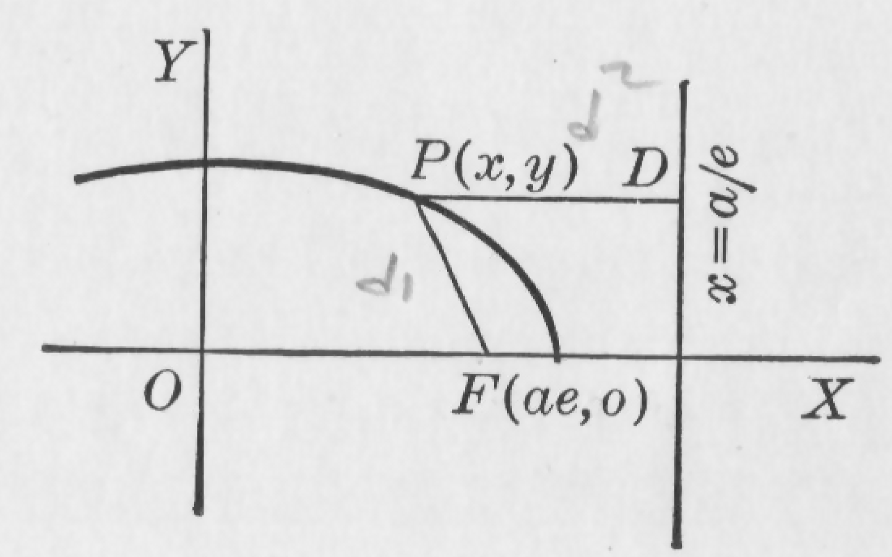


FIG. 58

$e = \frac{d_1}{d_2}$

Substituting in (2) and simplifying we obtain

$$(1 - e^2)x^2 + y^2 = a^2(1 - e^2), \tag{3}$$

which is equivalent to (1).

Thus if $P(x, y)$ is on the locus, (1) is satisfied. Furthermore if $P(x, y)$ is a point not on the locus, equation (2) is not true, and therefore equations (3) and (1) are not satisfied. Hence (1) is an equation of the locus.

From the equation $b^2 = a^2(1 - e^2)$, we find that

$$e^2 = \frac{a^2 - b^2}{a^2} = \frac{c^2}{a^2}.$$

Hence e is the eccentricity of the ellipse and F is a focus.

In the preceding theorem the point $F(ae, 0)$ and line $x = a/e$ may be replaced by the point $F'(-ae, 0)$ and the

line $x = -a/e$. Thus the ellipse has a *second directrix*, $x = -a/e$, corresponding to the focus F'.

50. Properties and applications of the ellipse. We state here without proof a few properties and applications of the ellipse.

(a) If a circle is projected by parallel lines from one plane on another, the projection is an ellipse (or a circle).

(b) The angle between the focal radii FP and $F'P$ to any point P on the ellipse is bisected by the line through P which is perpendicular to the tangent at P (this line is called the *normal*).

If an ellipse is rotated about an axis, an ellipsoidal surface is generated. Because of property (b), if rays of light or sound emanate from a focus F of an ellipsoidal reflecting surface they will all be reflected to the other focus F'. They will moreover take the same time to reach F'. This accounts for the phenomenon of "whispering galleries."

The earth and other planets move about the sun in orbits which are ellipses of small eccentricity (i.e., nearly circular orbits). Comets travel in orbits of larger eccentricity.

Arches in bridges and other structures are often elliptical.

In machinery elliptical gears are used when variable rates of motion are desired.

EXERCISES

1. Derive the equation of the locus of a point which moves so that the ratio of its distance from the point $F'(-ae, 0)$ to its distance from the line $x = -a/e$ is the constant e.

*Find the equation of each of the ellipses whose centers are at the origin, whose major axes are on the x-axis, and which satisfy the following conditions. Draw each curve showing foci, directrices, vertices, and latera recta.**

2. Eccentricity $= 1/10$, latus rectum $= 2$.
3. Focus is $(6, 0)$, eccentricity $= 4/5$.

* "Latera recta" is the plural of "latus rectum."

4. Directrix is $x = 13$, eccentricity $= 5/13$.
5. Directrix is $x = 26$, eccentricity $= 12/13$.
6. Major axis $= 24$, eccentricity $= 3/4$.
7. Minor axis $= 14$, directrix is $x = 64/\sqrt{15}$.

Find the foci, eccentricity, and directrices of each of the following ellipses.

8. $4x^2 + 9y^2 = 36$.
9. $9x^2 + 25y^2 = 225$.
10. $25x^2 + 169y^2 = 400$.
11. $144x^2 + 169y^2 = 900$.
12. $25x^2 + 4y^2 = 400$.
13. $100x^2 + 36y^2 = 900$.
14. $x^2 + y^2 = 100$.
15. $x^2 + .99^2y^2 = 1$.

THE HYPERBOLA

51. Standard equation of a hyperbola. *If a point moves in a plane so that the difference of its distances from two fixed points is a constant, the locus is a* **hyperbola.** The two fixed points are the **foci** of the hyperbola.

Let F' and F be the foci and P any point of a hyperbola. Denote the distance $\overline{F'F}$ by $2c$, and the difference of the distances $\overline{F'P}$ and $\overline{FP}$ (the larger less the smaller) by $2a$. Since the difference of two sides of a triangle is less than the third side,

$$2a < 2c \quad \text{or} \quad a < c.$$

By definition of a hyperbola

$$(1) \quad \overline{F'P} - \overline{FP} = \pm 2a,$$

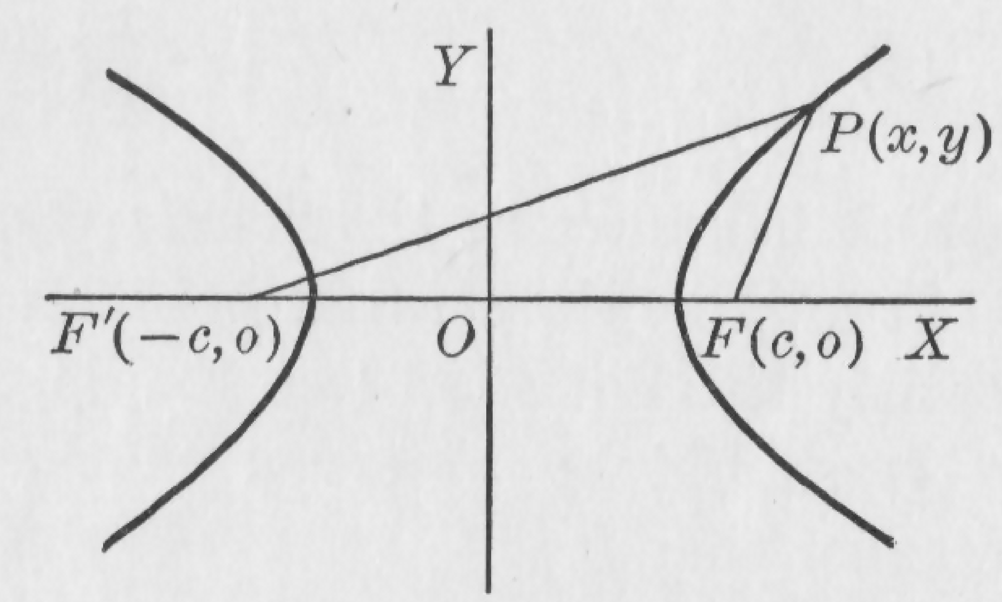


Fig. 59

the positive sign holding if $\overline{F'P} > \overline{FP}$, the negative sign if $\overline{F'P} < \overline{FP}$.

To obtain a simple equation for a hyperbola, let the line through F' and F be chosen as the x-axis, and the mid-point of $F'F$ as origin. The coördinates of F' are $(-c, 0)$, of F are $(c, 0)$, and of P are (x, y). Then from equation (1),

$$\sqrt{(x+c)^2 + y^2} - \sqrt{(x-c)^2 + y^2} = \pm 2a.$$

Hence

$$\sqrt{(x+c)^2+y^2} = \sqrt{(x-c)^2+y^2} \pm 2a.$$

Squaring and simplifying, we obtain

$$4cx = 4a^2 \pm 4a\sqrt{(x-c)^2+y^2},$$

$$cx - a^2 = \pm a\sqrt{(x-c)^2+y^2}.$$

If we square again and collect terms, we have

$$(c^2-a^2)x^2 - a^2y^2 = a^2c^2 - a^4.$$

Division by the right-hand member gives us

$$(2) \qquad \frac{x^2}{a^2} - \frac{y^2}{c^2-a^2} = 1.$$

Since $a < c$, the expression $c^2 - a^2$ is positive, and we may write for simplicity

$$(3) \qquad b^2 = c^2 - a^2;$$

then the equation becomes

$$(4) \qquad \frac{x^2}{a^2} - \frac{y^2}{b^2} = 1,$$

which is known as the **standard equation** for the hyperbola.

It may be shown that points not on the hyperbola do not satisfy this equation (see Ex. 33, p. 130).

It should be observed that a may be either larger or smaller than b.

If the foci are on the y-axis at $(0, c)$ and $(0, -c)$, the equation of the hyperbola is

$$(5) \qquad \frac{y^2}{a^2} - \frac{x^2}{b^2} = 1, \quad b^2 = c^2 - a^2.$$

52. Discussion of the hyperbola. It is geometrically obvious from the definition of a hyperbola that the curve is symmetrical with respect to the line through the foci F' and F, and to the perpendicular bisector of $F'F$. Their

point of intersection is a center of symmetry, bisecting every chord which passes through it, and is called the **center** of the hyperbola.

An analytic proof of the statements concerning symmetry would be similar to that given in § 47 for the ellipse.

To find the intercepts on the x-axis, substitute $y = 0$ in (4) and solve for x; we get $x = \pm a$. Thus the line passing through the foci crosses the curve at two points A' and A, the **vertices** of the hyperbola. The segment $A'A$ is called the **transverse axis**; its length is $2a$.

If we substitute $x = 0$ in equation (4) we find for y the imaginary values

$$y = \pm b\sqrt{-1};$$

hence the y-axis does not cross the curve. For a reason which will be obvious later (see § 54) the line from $B'(0, -b)$ to $B(0, b)$ is called the **conjugate axis**; its length is $2b$.

The chord through a focus perpendicular to the transverse axis is the **latus rectum** of the hyperbola (LR in Fig. 60). To find its length, substitute $x = c$ in equation (4) and solve for y; we find

$$y^2 = b^2\,\frac{c^2-a^2}{a^2} = \frac{b^4}{a^2},$$

$$y = \pm\frac{b^2}{a}.$$

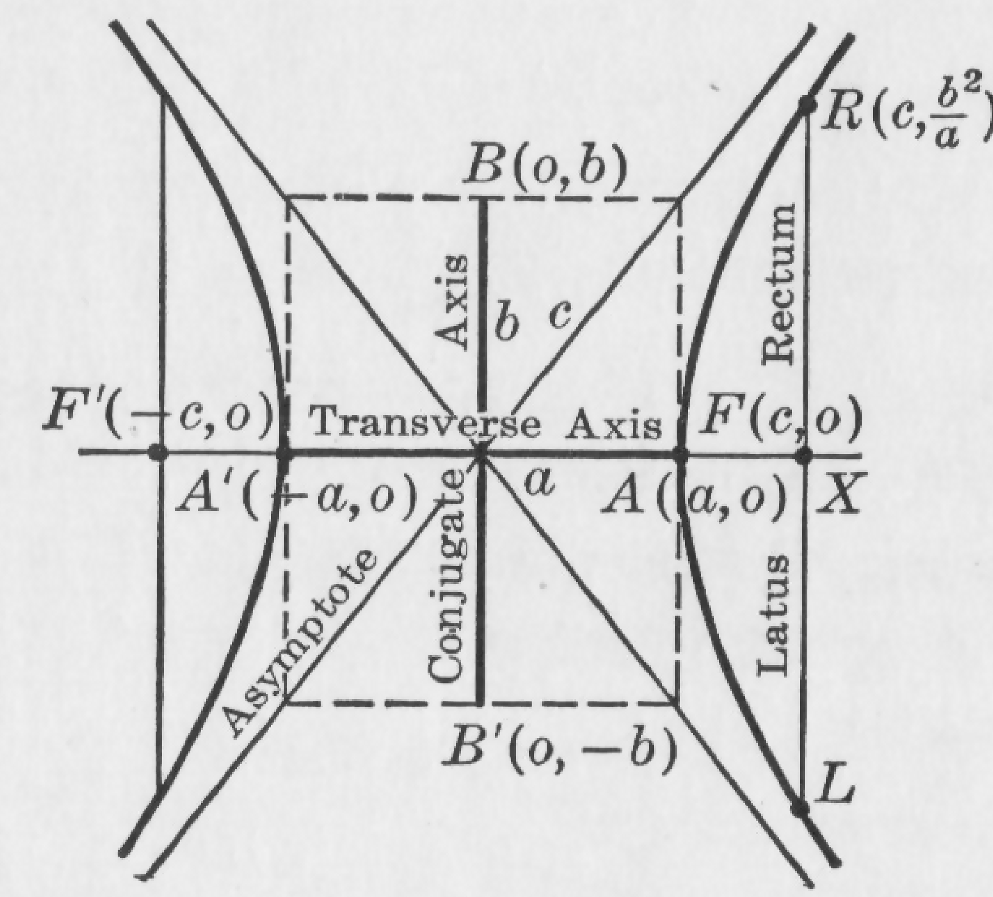


FIG. 60

Thus $FR = b^2/a$, and the length of the latus rectum is $2b^2/a$, the same expression as was found for the latus rectum of an ellipse in § 47.

Solving equation (4) for y we obtain

$$y = \pm\frac{b}{a}\sqrt{x^2-a^2}.$$

If x is numerically less than a, the corresponding value of y is imaginary. Hence there is no portion of the hyperbola between the lines $x = a$ and $x = -a$.

Consider for a moment that part of the curve for which y is positive and x is as great as a. It is easily seen that if we give x any series of increasing values, such as a, $2a$, $3a$, $\cdots$, the corresponding values of y increase constantly. Furthermore for a large value of x, such as $x = 10a$, the square root of $x^2 - a^2$ is approximately equal to x, and hence $y = bx/a$ approximately. It is proved in the next article that the distance from the curve to the line

$$y = \frac{bx}{a}$$

approaches zero as points are taken farther and farther out on the curve. This line is called an **asymptote** of the hyperbola. The line

$$y = -\frac{bx}{a}$$

is also an asymptote. The two asymptotes are given by the single equation

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 0,$$

as may be seen by solving this equation for y.

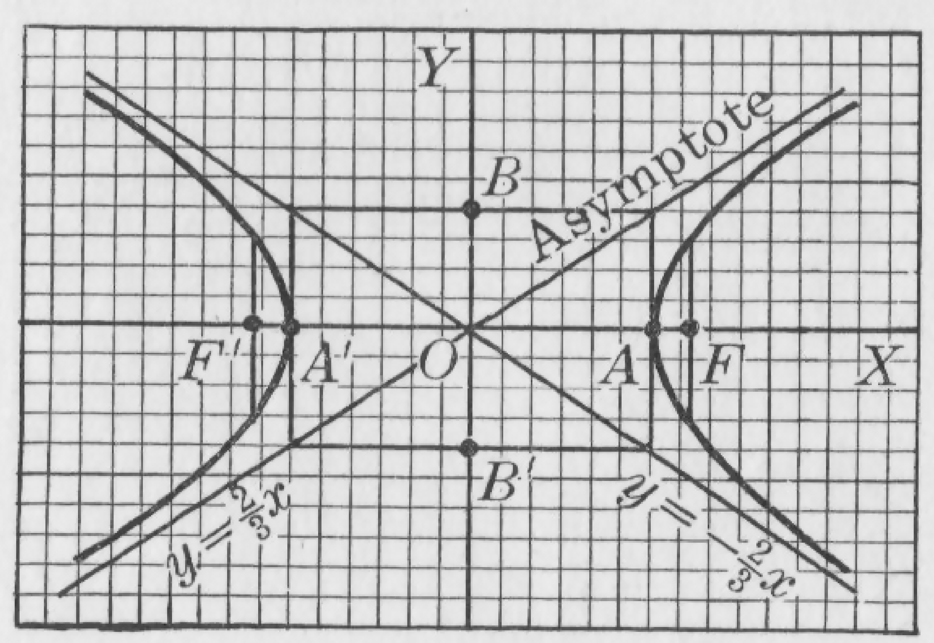


FIG. 61

Example. — Draw the curve whose equation is

$$\frac{x^2}{36} - \frac{y^2}{16} = 1.$$

Solution. — The equation has the standard form (4) of a hyperbola. We have

$$a = 6, \quad b = 4, \quad c = \sqrt{36 + 16} = 2\sqrt{13}.$$

The vertices are $(\pm 6, 0)$, the foci $(\pm 2\sqrt{13}, 0)$. The length of the latus rectum is $32/6 = 16/3$. The asymptotes are the lines

$$y = \pm \tfrac{2}{3}x.$$

After locating the vertices and the ends of the latera recta, and drawing the asymptotes, we easily sketch the curve, as in Figure 61.

EXERCISES

1. Derive the equation (5), of page 126, for the hyperbola whose foci are $(0, \pm c)$.

2. Find the equations of the asymptotes of the hyperbola of Exercise 1.

3. Show that the points $Q(a, b)$ and $Q'(b, a)$ lie on asymptotes of the respective hyperbolas

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \quad \text{and} \quad \frac{y^2}{a^2} - \frac{x^2}{b^2} = 1.$$

4. Show that the circle which has its center at the origin and which passes through $Q(a, b)$ also passes through the foci of the hyperbolas of Exercise 3.

Write each of the following equations in form (4) or (5), § 51. Find the foci, vertices, length of latus rectum, and equations of the asymptotes. Draw the curves.

5. $4x^2 - 9y^2 = 36.$ **6.** $9x^2 - 4y^2 = 36.$

7. $x^2 - y^2 = 36.$ **8.** $x^2 - y^2 = 64.$

9. $x^2 - 25y^2 = 100.$ **10.** $25x^2 - y^2 = 100.$

11. $64x^2 - y^2 = 81.$ **12.** $x^2 - 49y^2 = 100.$

13. $4x^2 - 9y^2 = -36.$ **14.** $9x^2 - 4y^2 = -36.$

15. $x^2 - y^2 = -36.$ **16.** $x^2 - y^2 = -64.$

17. $x^2 - 25y^2 = -100.$ **18.** $25x^2 - y^2 = -100.$

19. $64x^2 - y^2 = -81.$ **20.** $x^2 - 49y^2 = -100.$

Find an equation of each of the hyperbolas whose centers are at the origin, whose transverse axes are on the x-axis, and which fulfil the following conditions.

21. A focus is at (10, 0), a vertex at (6, 0).

22. A focus is at (13, 0), a vertex at (5, 0).

23. A focus is at (5, 0), transverse axis = 6.

24. A focus is at (5, 0), transverse axis = 8.

25. Transverse axis = 14, latus rectum = 14.

26. Transverse axis = 8, latus rectum = 32.

27. It passes through the points (7, 4) and (− 4, 2).

28. It passes through the points (6, 2) and (− 5, 1).

Find an equation of each of the hyperbolas whose centers are at the origin, whose conjugate axes are on the x-*axis, and which fulfil the conditions given in Exercises* 29–32.

29. Distance between the foci is 10, between the vertices 8.

30. Latus rectum = 9; an asymptote is $y = \frac{2}{3}x$.

31. Transverse axis = 8; and the curve passes through the point $(3\sqrt{7}, 8)$.

32. It passes through (3, 5); the asymptotes are mutually perpendicular. $a^2 = b^2$

33. Prove that a point $P(x, y)$ not on the hyperbola defined in § 51 does not satisfy the equation $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$.

Hint. If P is not on the hyperbola the difference of the focal radii $\overline{F'P}$ and $\overline{FP}$ is $2a'$ where $a' \neq a$. Then P satisfies the equation

$$\frac{x^2}{a'^2} - \frac{y^2}{b'^2} = 1, \text{ where } b'^2 = c^2 - a'^2.$$

Show that this equation is incompatible with the equation

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1.$$

53. Asymptotes. If a point in moving continuously along a curve recedes indefinitely far from the origin, and at the same time approaches indefinitely near to a given straight line, the line is called an **asymptote** of the curve.

Let us prove that the lines

$$(1) \qquad \frac{x}{a} - \frac{y}{b} = 0, \quad \frac{x}{a} + \frac{y}{b} = 0,$$

are asymptotes of the hyperbola

$$(2) \qquad \frac{x^2}{a^2} - \frac{y^2}{b^2} = 1.$$

From considerations of symmetry it is seen that it will suffice to show that if a point $P_1(x_1, y_1)$ in the first quadrant moves out indefinitely far along the curve (2) its distance d from the line

$$\frac{x}{a} - \frac{y}{b} = 0$$

approaches zero as a limit.

We have

$$d = \frac{bx_1 - ay_1}{\sqrt{a^2 + b^2}}.$$

Since P_1 is on the curve (2),

$$b^2x_1^2 - a^2y_1^2 = a^2b^2,$$

and hence

$$bx_1 - ay_1 = \frac{a^2b^2}{bx_1 + ay_1}.$$

It follows that

$$d = \frac{a^2b^2}{\sqrt{a^2 + b^2}(bx_1 + ay_1)}.$$

As $P_1(x_1, y_1)$ recedes indefinitely far, x_1 and y_1 increase indefinitely and therefore d approaches zero as a limit.

54. Conjugate hyperbolas. Consider the two hyperbolas

$$(1) \qquad \frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \quad \text{and} \quad \frac{x^2}{a^2} - \frac{y^2}{b^2} = -1.$$

It is observed that the transverse axis of the first, extending from $(-a, 0)$ to $(a, 0)$, is the conjugate axis of the second; also the conjugate axis of the first is the transverse axis of the second. Moreover the distance from the center to a focus of the first is $\sqrt{a^2 + b^2}$, and to a focus of the second is $\sqrt{b^2 + a^2}$; thus this distance is the same for the two hyperbolas. Furthermore the asymptotes are the same for the two hyperbolas; for their equations can be written, in each case,

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 0.$$

The two hyperbolas thus closely related are called **conjugate hyperbolas.**

The graphs of the conjugate hyperbolas

$$\frac{x^2}{16} - \frac{y^2}{9} = 1 \quad \text{and} \quad \frac{x^2}{16} - \frac{y^2}{9} = -1,$$

are shown in the adjacent figure.

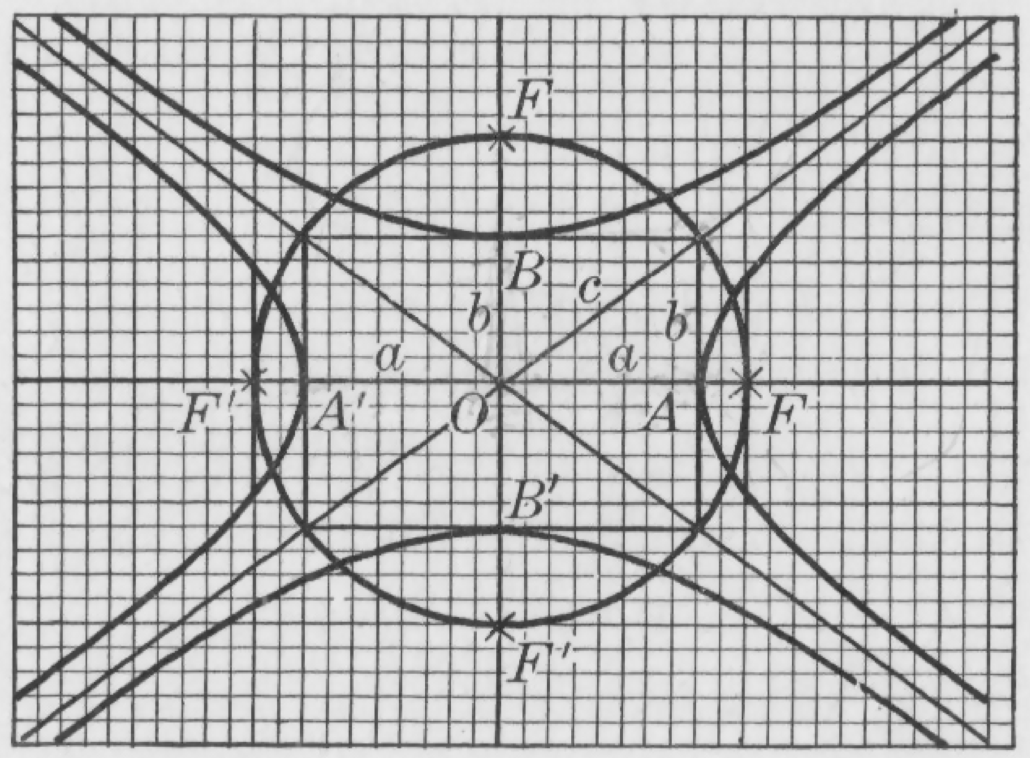


FIG. 62

55. Eccentricity of a hyperbola. The ratio c/a is called the **eccentricity** of a hyperbola, and is denoted by e,

$$e = c/a.$$

Since $c^2 = a^2 + b^2$, it follows that $c > a$; hence for a hyperbola $e > 1$.

If b is very small compared with a, then c is only slightly larger than a, and the eccentricity is but little greater than 1. In this case the asymptotes make a small angle with the transverse axis, and the hyperbola is sharply curved at its vertices.

If e is very large, we find by a similar argument that the curve is rather flat at the vertices.

56. Directrix of a hyperbola. An important property of the hyperbola is given by the following theorem:

If a point moves so that the ratio of its distance from the point F(ae, 0) *to its distance from the line* x = a/e *is always equal to the constant* e *greater than unity, the locus is the hyperbola*

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1, \quad where \quad b^2 = a^2(e^2 - 1).$$

The point F *is the focus of the hyperbola and the constant* e *is the eccentricity.*

The line $x = a/e$ is called the **directrix** of the hyperbola corresponding to the focus F.

The proof of the theorem is similar to that of the theorem in § 49. In this case $e > 1$ so that we have $b^2 = a^2(e^2 - 1)$ instead of $b^2 = a^2(1 - e^2)$; this changes a sign in each of two equations. The student should go through the details.

In the preceding theorem the point $F(ae, 0)$ and the line $x = a/e$ may be replaced by the point $F'(-ae, 0)$ and the line $x = -a/e$. The latter line is a directrix corresponding to the focus F'.

57. Applications of the hyperbola. If one variable y varies inversely as a second variable x, then $xy = k$, where k is a constant. This is the equation of a hyperbola (see page 149), the coördinate axes being the asymptotes. Boyle's Law in physics gives an example of such variation; it may be stated thus: For a perfect gas the pressure p varies inversely as the volume v of a given mass of the gas.

If a comet approaches the sun with sufficiently high velocity its orbit is a branch of a hyperbola with the sun at a focus.

During the World War the position of a hidden gun was detected by use of the following method. Instruments for recording sound were located at two points F' and F. The report of a gun at P was heard at F, let us say, 2 seconds earlier than at F'. Since sound travels at approximately 1087 feet per second, it follows that

$$\overline{F'P} - \overline{FP} = 2 \times 1087 = 2174 \text{ ft.}$$

Hence P was on the hyperbola whose foci were at F' and F and whose transverse axis was of length 2174 feet; it was on the branch nearer F than F'. By using a second pair of recording instruments at F_1' and F_1, P was located on a second hyperbola. The intersection of the two hyperbolas gave the position of the gun. It is said that a gun 10 miles distant could be located within 50 feet by this method when carried out with all possible accuracy.

EXERCISES

1. Prove directly that a point in the second quadrant moving out indefinitely along the hyperbola $b^2x^2 - a^2y^2 = a^2b^2$ approaches indefinitely near to the line $bx + ay = 0$.

In each of the Exercises 2–5 draw the graph of the hyperbola and of its conjugate, showing foci, vertices, axes, asymptotes, and latera recta.

2. $25x^2 - 9y^2 = 144$. **3.** $36x^2 - 4y^2 = 225$.

4. $x^2 - 144y^2 + 400 = 0$. **5.** $x^2 - y^2 + 400 = 0$.

6. Find the eccentricity and the directrices of each of the hyperbolas given by the equations of Exercises 2, 3, 4, 5.

7. Given a focus F at a distance p from the corresponding directrix of a hyperbola of given eccentricity e greater than unity, find in terms of p and e the position of coördinate axes such that the coördinates of F are $(ae, 0)$ and the equation of the directrix is $x = a/e$.

8. A **rectangular hyperbola** is one whose asymptotes intersect at right angles. An equilateral hyperbola is one whose transverse and conjugate axes are equal in length. Show that the equation of a rectangular hyperbola whose center is at the origin and whose transverse axis lies on the x-axis is $x^2 - y^2 = a^2$. Show that it is equilateral.

9. Prove that the eccentricity of a rectangular hyperbola (Ex. 8) is $\sqrt{2}$.

10. Prove that if e_1 and e_2 are the eccentricities of a hyperbola and its conjugate, then $\frac{1}{e_1^2} + \frac{1}{e_2^2} = 1$.

11. Let Q be the foot of a perpendicular from a focus F to an asymptote of a hyperbola. If the length of the transverse axis is $2a$ and that of the conjugate axis is $2b$, show that the length of FQ is b, and that the distance from the center of the hyperbola to Q is a.

CONSTRUCTIONS FOR THE CONICS

★ 58. Constructions for the parabola. (1) *A point by point construction* of a parabola is readily made on coördinate paper as follows. Let the directrix MN be a line of the ruled paper, and let the focus F be on a ruling. Draw the line DF perpendicular to MN. Let R be any point on DF, or DF produced, whose distance from F is not greater than its distance from D. With F as center and radius DR draw a circle; it will cut the ruled line through R parallel to the directrix in two points P and P'. These points are at the same distance from the directrix as from the focus, and therefore lie on the parabola. By varying the choice of R, we obtain as many points as we desire on the parabola.

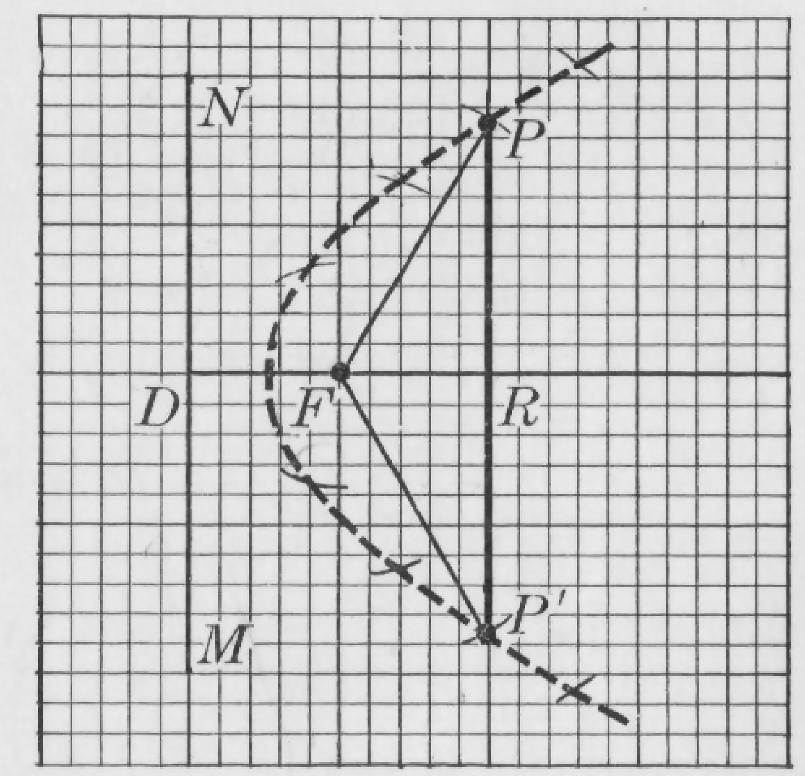


FIG. 63

(2) *Triangle and thread construction.* Place a draftsman's triangle so that it may slide along a ruler, with a side of the triangle perpendicular to the ruler (Fig. 64). Fasten one end of a string of length QS at S on the triangle, the other at F (using a thumbtack). With a pencil draw the string taut against the side of the triangle. Slide the triangle, tracing a curve with the pencil P. This curve is a parabola with focus at F and directrix along the ruler's edge.

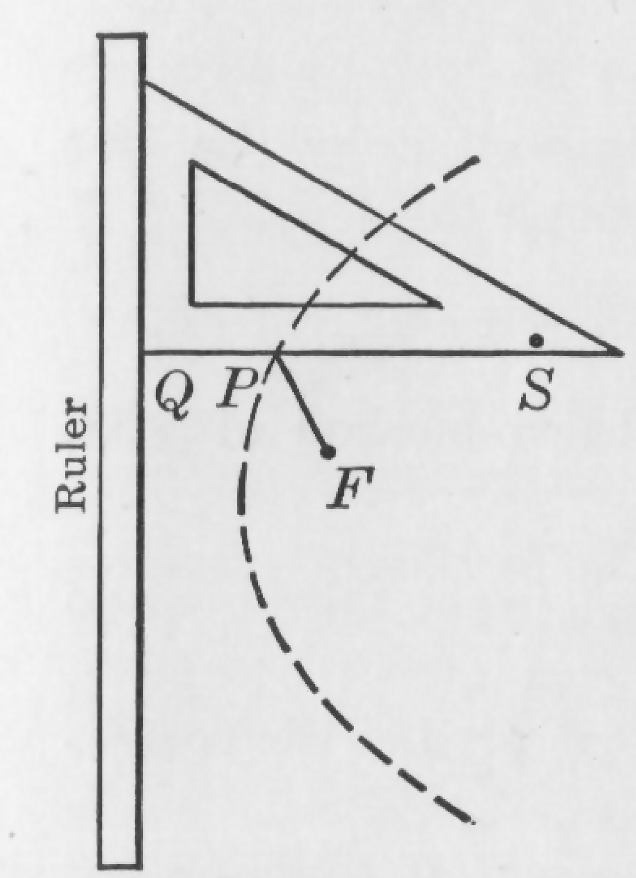


FIG. 64

(3) *Parabolic arch.* Suppose the span BC of a parabolic arch and its height h are given. It is required to find points on the arch. Let A be the mid-point of BC; draw $AO = h$

perpendicular to BC. Construct the rectangle $BCDE$ of altitude h. Divide BA and BE into the same number of equal parts, for example, five. Let the points of division be a, b, c, d and a', b', c', d' as shown in the Figure. Draw

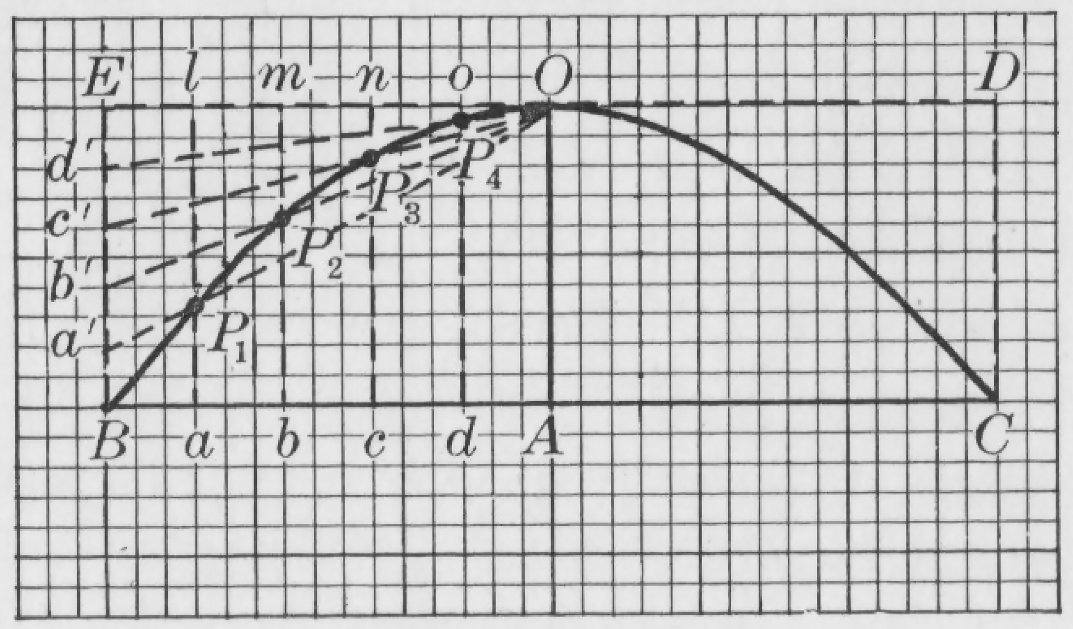


FIG. 65

al parallel to AO; and draw Oa' intersecting al at P_1. Similarly draw bm and Ob' intersecting at P_2, etc. The points $P_1, P_2, \cdots$ lie on the parabolic arch.

EXERCISES

1. Construct a parabola by locating ten points on it by the first method of § 58, and drawing a smooth curve through them.

2. Construct a parabola by the second method of § 58.

3. Construct a large parabolic arch by the third method of § 58, locating twenty points on the curve.

4. Prove that the points $P_1, P_2, \cdots$ mentioned in the third method of § 58 lie on a parabola which passes through the points B, C, and O.

Hint. Take O as origin, OA as x-axis; let $AB = q$. Let the coördinates of one of the points P be (x, y). Show that $y^2 = \frac{q^2}{h}x$.

★ 59. Constructions for the ellipse. (1) *Location of points by compass.* We assume that the foci F' and F, and the distance $2a$ are given. Draw MN of length $2a$. Take a point R_1 on MN (Fig. 66). Draw a circle with F' as center

and radius equal to MR_1, and another circle with F as center and radius equal to NR_1. The circles intersect at points which lie on the ellipse. By taking a succession of points $R_1, R_2, \cdots$ on MN we obtain a set of points on the ellipse.

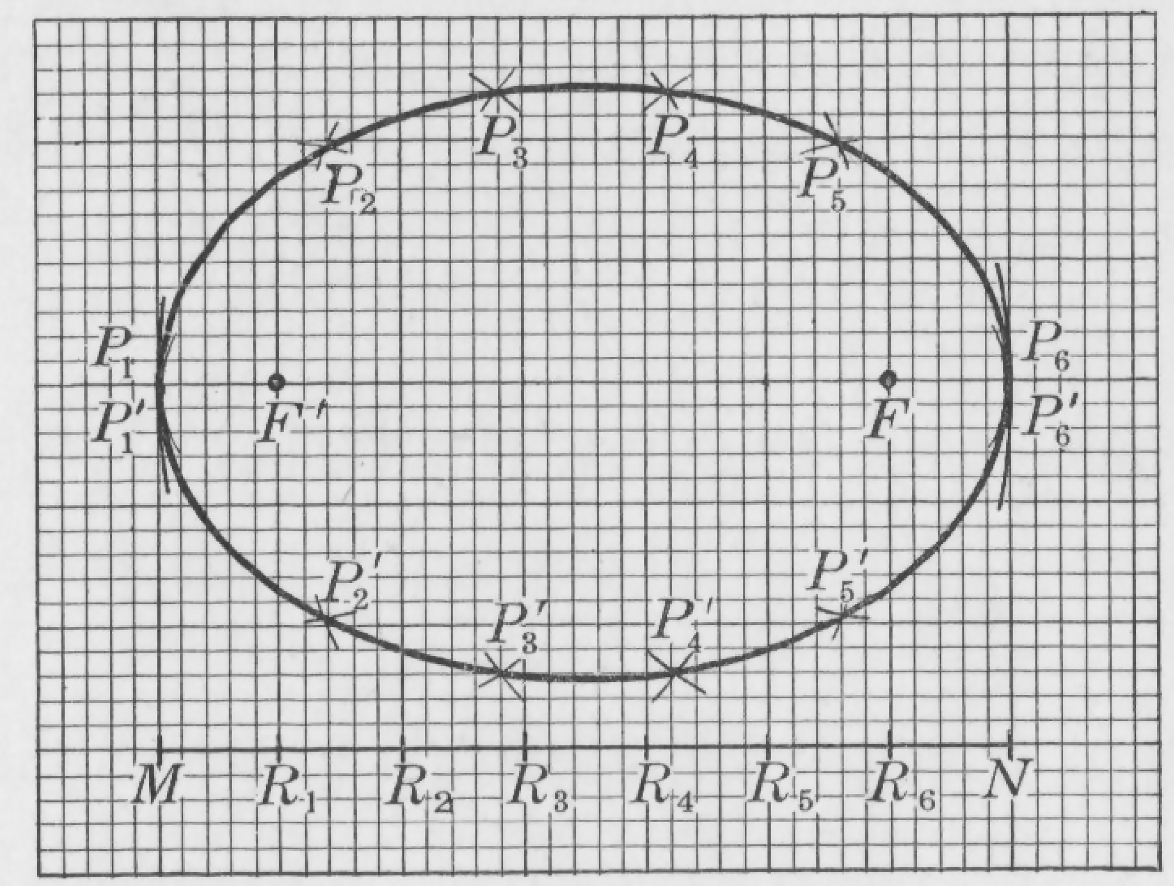


FIG. 66

(2) *A thread construction.* Place thumb-tacks at the foci. Form a loop of thread of length $2a + 2c$, to which a pencil is attached at a point P. Draw the thread taut with the pencil so that the tacks at the foci are inside the loop, and while keeping the thread taut describe a curve. It is an ellipse.

(3) *Construction by aid of auxiliary circles.* Given the axes $A'A$ and $B'B$ of an ellipse, draw circles with these lines as diameters. Draw a radial line from the center O intersecting these circles in points which we shall call M and N (Fig. 67). The line through M parallel to the minor axis intersects the line through N parallel to the major axis in a point P which is on the ellipse. By taking a series of radial lines and pro-

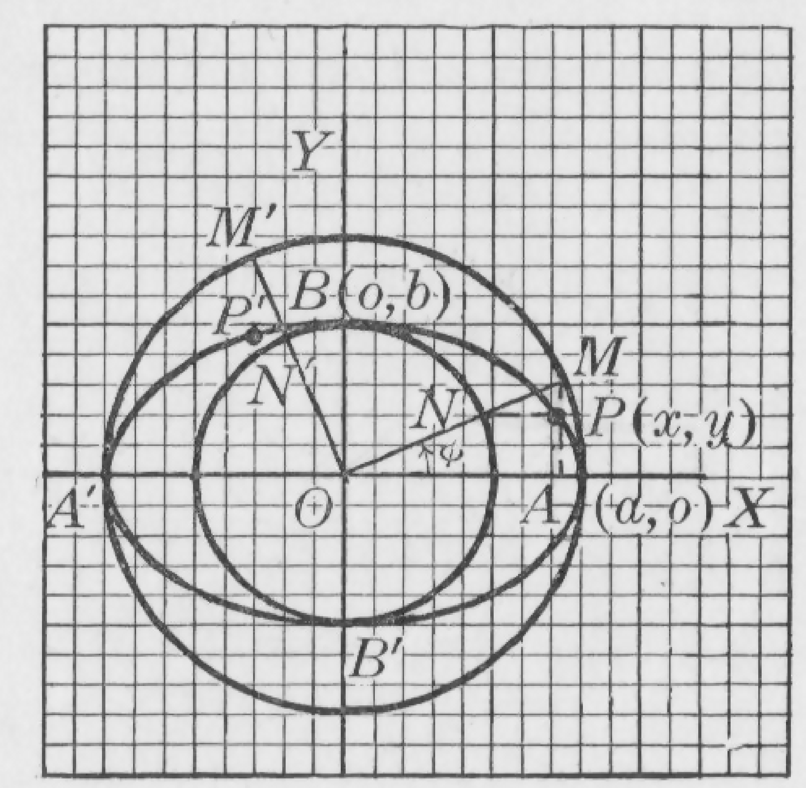


FIG. 67

ceeding in this way we locate as many points as desired on the ellipse.

The two circles drawn in this construction are called the **auxiliary circles** of the ellipse. The larger is the **major auxiliary circle,** the smaller the **minor auxiliary circle.**

To prove that P lies on the ellipse, we observe that in the notation of the figure, since $OM = a$, $ON = b$, we have

$$\frac{x}{a} = \cos\phi, \qquad \frac{y}{b} = \sin\phi.$$

Hence, by squaring and adding, we obtain

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$$

The angle ϕ is called the **eccentric angle** of the point P.

EXERCISES

1. Draw a large ellipse by the first method of § 59, using twenty points R_1, R_2, $\cdots$. Take $F'F$ nearly equal to MN.

2. Draw two ellipses by the second method of § 59, taking first $a = 2c$ and second $a = 4c$.

3. Draw a large ellipse by the third method of § 59, taking $a = 3b$.

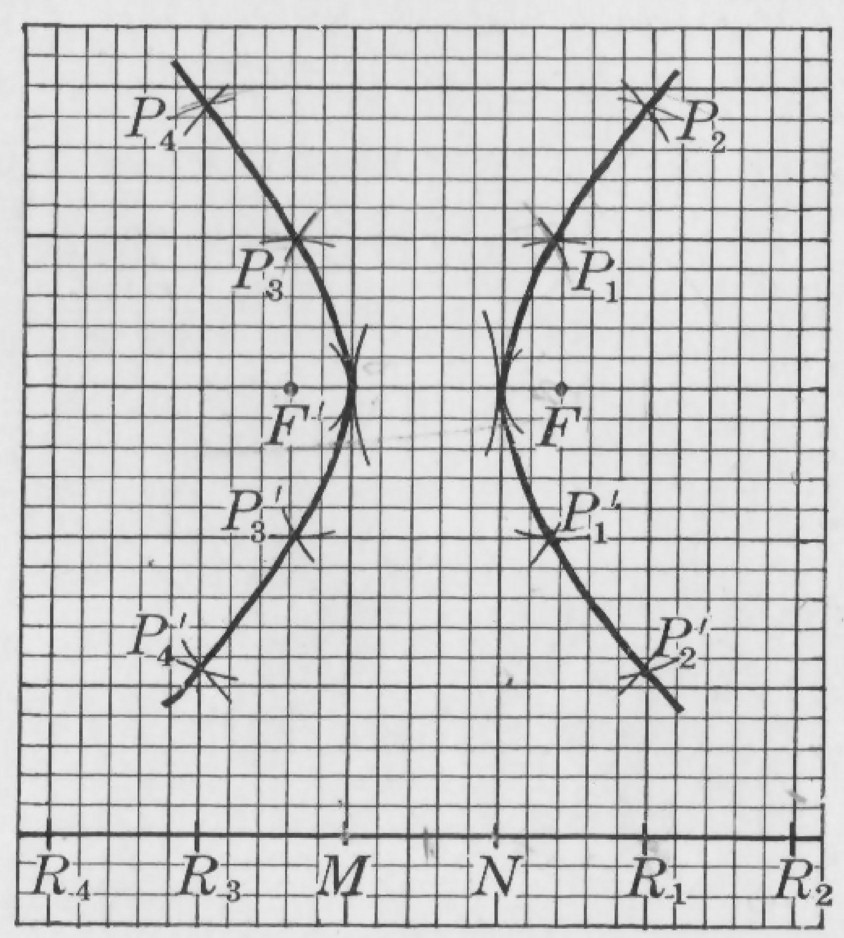


FIG. 68

★ 60. Constructions for the hyperbola. (1) *Location of points by compass.* We assume that the foci F' and F and the transverse axis are given. Draw MN of length $2a$. On MN produced take a point R_1 (Fig. 68). With F' as center and MR_1 as radius, and with F as center and NR_1 as radius draw circles, which intersect in points P_1' and P_1. These points are on the hyperbola. By taking a series of

points R_1, R_2, $\cdot\cdot\cdot$, and, proceeding similarly, we obtain further points on the hyperbola.

(2) *A thread construction.* Place thumb-tacks at the foci. Pass over F' and around F a thread whose ends are held together (Fig. 69). Fasten a pencil at a point P of the thread, pull it taut, and trace a curve by allowing the thread to slide past F' and F. The curve described is a hyperbola, since PF' and PF increase equally during the motion and hence $\overline{PF'} - \overline{PF}$ is constant.

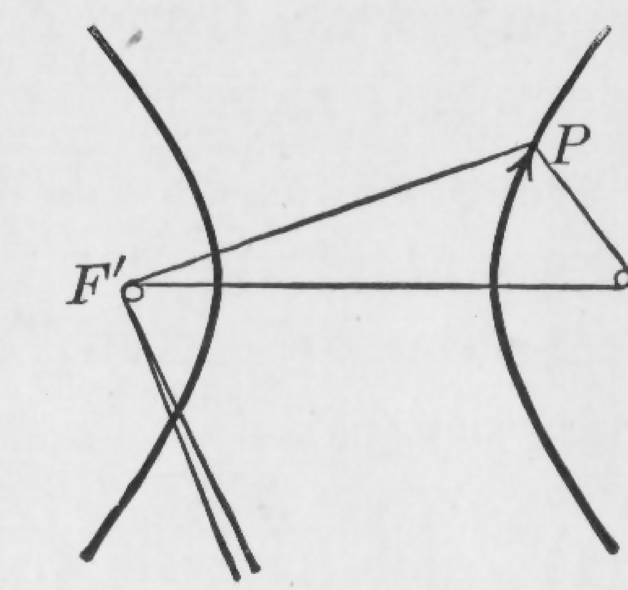


FIG. 69

EXERCISES

1. Draw a hyperbola by the first method of § 60.

2. Draw a hyperbola by the second method of § 60.

POLAR COÖRDINATES

★ 61. A general definition of a conic. In order to derive a polar equation of a conic in the simplest form we first note that the definition of a parabola (§ 43) and the theorems of §§ 49 and 56 may be combined in the following statement:

A conic is the locus of a point which moves in a plane so that the ratio of its distance from a given point (a focus) *to its distance from a given line* (a directrix) *is a constant,* e (the eccentricity). *If*

e $<$ 1, *the conic is an ellipse;*
e $=$ 1, *the conic is a parabola;*
e $>$ 1, *the conic is a hyperbola.*

The preceding statement may be taken as a new definition of a conic. It is here assumed that the focus does not lie on the directrix and that the focus and the directrix are in the finite plane. Limiting forms are obtained if any of these conditions are not fulfilled.

★ 62. The conic in polar coördinates. We shall use the definition of a conic given in § 61. Let F be a focus and MN the corresponding directrix. Choose F as the pole and the perpendicular from F to MN as the polar axis of a system of polar coördinates. Let $P(r, \theta)$ be any point of the conic and PQ the perpendicular from P to MN. Then, by § 61,

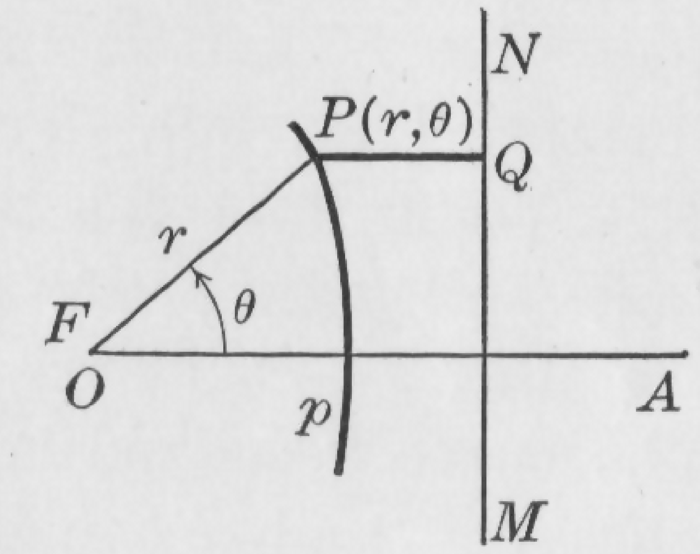


FIG. 70

$$(1) \qquad \frac{\overline{FP}}{\overline{PQ}} = e.$$

It is seen that $\overline{FP} = r$. Letting the distance from the focus to the directrix be p, we have $\overline{PQ} = p - r \cos \theta$. Hence from (1),

$$r = ep - er \cos \theta;$$

it follows that *

$$(2) \qquad r = \frac{ep}{1 + e \cos \theta};$$

$e < 1$, the curve is an ellipse;
$e = 1$, the curve is a parabola;
$e > 1$, the curve is a hyperbola.

The equation of the directrix is $r \cos \theta = p$.

If the pole is a focus of a conic, and if a polar equation of the corresponding directrix is

$$r \cos (\theta - \alpha) = p,$$

then, as the student may show, a polar equation of the conic is

$$(3) \qquad r = \frac{ep}{1 + e \cos (\theta - \alpha)}.$$

* Equation (2) was derived on the assumption that F and P lie to the left of MN, and that r is positive. It can be shown, however, that if r is allowed to have negative values, then (2) is satisfied by *some* coördinates of each point on the conic, and by *no* coördinates of any point not on the conic. It is therefore an equation of the conic.

EXERCISES

1. Show that the polar equation (2), § 62, may be written, for the case of a parabola, in the form

$$r = \frac{p}{2} \sec^2 \frac{\theta}{2}.$$

Determine the eccentricity, and the equation of the directrix, for each of the following conics, and plot the curve.

2. $r = \dfrac{16}{1 + 2 \cos \theta}$. **3.** $r = \dfrac{8}{2 + \cos \theta}$.

4. $r = \dfrac{8}{1 + \cos \theta}$. **5.** $r = \dfrac{5}{3 + 4 \cos \theta}$.

6. $r = \dfrac{8}{1 - \cos \theta}$. **7.** $r = \dfrac{12}{3 - 5 \cos \theta}$.

8. $r = \dfrac{10}{5 - \cos \theta}$. **9.** $r = \dfrac{8}{4 - 3 \cos \theta}$.

10. Show that the center of the conic (2), § 62, when e is not unity, is the point $\left(\dfrac{e^2 p}{e^2 - 1}, 0\right)$.

11. Show that a vertex of the conic (2), § 62, when e is not unity, is the point $\left(\dfrac{ep}{1 + e}, 0\right)$, and that another vertex is the point $\left(\dfrac{ep}{e - 1}, 0\right)$.

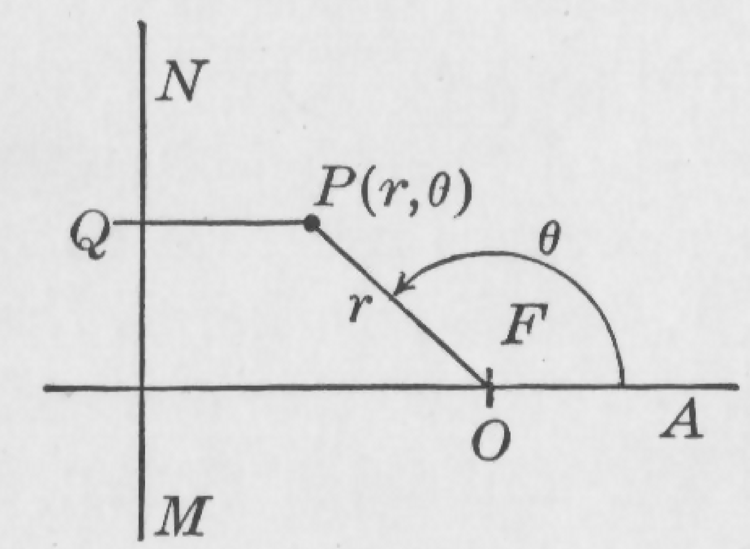


FIG. 71

12. Prove that if the positive direction of the polar axis is away from the directrix (Fig. 71), then the equation of the conic is

$$r = \frac{ep}{1 - e \cos \theta}.$$

13. Prove formula (3), § 62, by use of a figure.

CHAPTER VII

TRANSFORMATION OF RECTANGULAR COÖRDINATES

63. Change of axes. It is often an important matter to be able to solve the following problem: Given the equation of a curve with respect to one set of coördinate axes, to find the equation with respect to another set of axes. The operation of changing from one set to the other, called a **change of axes,** corresponds to a **transformation of coördinates,** whose formulas express the old coördinates of a point in terms of the new coördinates.

By changing the axes we may be able to transform an equation in the old coördinates into a simpler one in the new system. This is illustrated in the following sections. In Chapter XII the general equation of second degree is reduced to standard forms by this method.

64. Translation of axes. If the new axes $O'X'$ and $O'Y'$ are parallel to the old axes OX and OY with positive directions on the axes preserved, the axes are said to be *translated* from one position to the other.

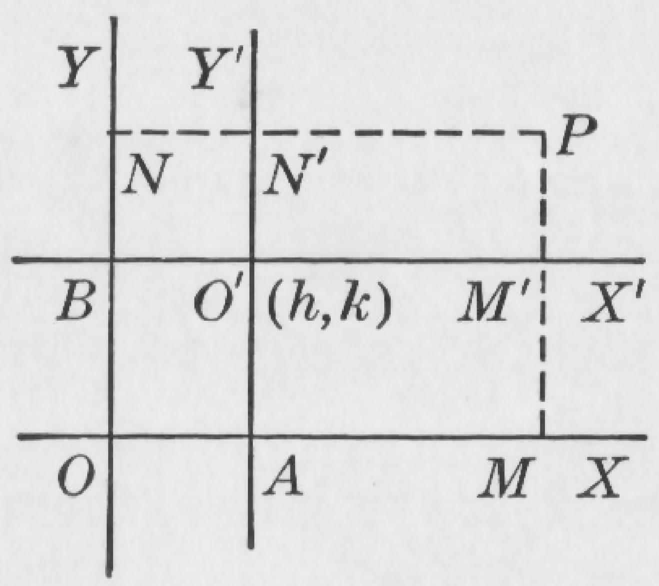


FIG. 72

Let the new origin have the coördinates (h, k) with reference to the old axes. Let P be any point in the plane. Its coördinates are (x, y) with respect to the old axes and (x', y') with respect to the new axes, where (Fig. 72)

$$x = OM, \qquad y = ON;$$
$$x' = O'M', \quad y' = O'N'.$$

Since

$$h = OA, \quad k = OB,$$

we have at once

$$(1) \qquad \boldsymbol{x = x' + h, \qquad y = y' + k.}$$

These are the *equations for translating the axes* to the new origin (h, k).

Example. — The equation of a curve with respect to one set of axes is

$$x^2 + 4y^2 + 4x - 8y = 8.$$

Translate the axes to the new origin $(-2, 1)$, find the equation in the new coördinates, and draw the graph of the equation.

Solution. — We have, from (1),

$$x = x' - 2, \quad y = y' + 1;$$

the equation becomes

$$(x' - 2)^2 + 4(y' + 1)^2 + 4(x' - 2) - 8(y' + 1) = 8.$$

Simplifying, we obtain

$$x'^2 + 4y'^2 = 16.$$

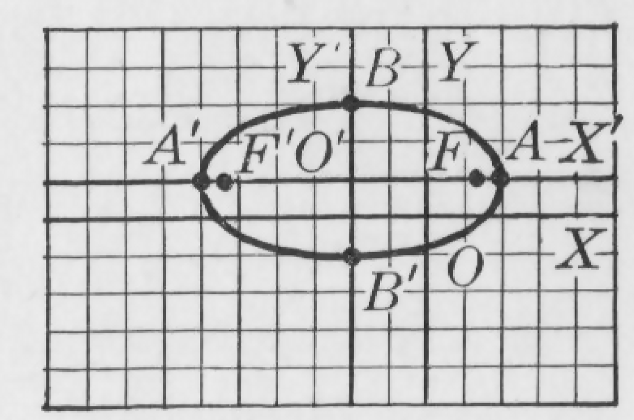


FIG. 73

This new equation is recognized as the equation of an ellipse. By drawing its graph with respect to the new axes we obtain the graph of the first equation with respect to the old axes. With respect to the new axes the coördinates of the foci are

$$x' = \sqrt{12}, \quad y' = 0, \quad \text{and} \quad x' = -\sqrt{12}, \quad y' = 0;$$

hence with respect to the old axes the coördinates are

$$x = \sqrt{12} - 2, \quad y = 1, \quad \text{and} \quad x = -\sqrt{12} - 2, \quad y = 1.$$

EXERCISES

Find the coördinates of the following points after translating the axes as indicated. Draw a figure and verify the results.

1. New origin $O'(2, 5)$. Points: $A(6, 8)$, $B(-1, 2)$, $C(0, 0)$.

2. New origin $O'(-2, 3)$. Points: $A(4, 4)$, $B(-4, 2)$, $C(0, 0)$.

Transform each of the following equations when the axes are translated as indicated. Plot both pairs of axes and the curve.

3. $x^2 + y^2 + 6x - 8y = 0$; $O'(-3, 4)$.

4. $x^2 + y^2 - 8x + 6y = 24$; $O'(4, -3)$.

5. $(x - h)^2 + (y - k)^2 = r^2$; $O'(h, k)$.

6. $(y - k)^2 = 2p(x - h)$; $O'(h, k)$.

7. $y^2 - 4y - 6x - 8 = 0$; $O'(-2, 2)$.

8. $x^2 + 4x + 12y = 8$; $O'(-2, 1)$.

9. $4x^2 + 9y^2 + 8x - 18y = 3$; $O'(-1, 1)$.

10. $16x^2 + y^2 + 64x = 0$; $O'(-2, 0)$.

11. $16x^2 + 9y^2 - 18y = 40$; $O'(0, 1)$.

12. $4x^2 - 9y^2 + 16x = 20$; $O'(-2, 0)$.

13. $9x^2 - 4y^2 - 24y = 72$; $O'(0, -3)$.

14. $16x^2 - y^2 - 32x + 8y = 100$; $O'(1, 4)$.

65. Applications to conic sections. Equations of the conics were found in Chapter VI as follows:

$$\textit{Parabola,}\quad y^2 = \pm 2px, \quad \text{or} \quad x^2 = \pm 2py.$$

$$\textit{Ellipse,}\quad \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, \quad \text{or} \quad \frac{y^2}{a^2} + \frac{x^2}{b^2} = 1.$$

$$\textit{Hyperbola,}\quad \frac{x^2}{a^2} - \frac{y^2}{b^2} = 1, \quad \text{or} \quad \frac{y^2}{a^2} - \frac{x^2}{b^2} = 1.$$

What equations will be reduced by the translation of axes

$$(1) \qquad x = x' + h, \quad y = y' + k$$

to equations of these types in x' and y'? The answer is found by making the inverse substitution

$$x' = x - h, \quad y' = y - k.$$

Thus to get, for example,

$$y'^2 = 2px',$$

the original equation must be equivalent to

$$(y - k)^2 = 2p(x - h).$$

Moreover every equation of this last form can be reduced to the preceding standard form by a substitution (1).

The general results are as follows:

An equation is that of a parabola with its axis parallel to a coördinate axis if and only if it can be written in one of the forms

$$(2) \qquad (y - k)^2 = \pm 2p(x - h) \text{ or } (x - h)^2 = \pm 2p(y - k).$$

The vertex is the point (h, k).

An equation is that of an ellipse with axes parallel to the coördinate axes if and only if it can be written in one of the forms

$$(3) \qquad \frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1, \quad \frac{(x-h)^2}{b^2} + \frac{(y-k)^2}{a^2} = 1.$$

The center of the ellipse is the point (h, k).

An equation is that of a hyperbola with axes parallel to the coördinate axes if and only if it can be written in one of the forms

$$(4) \qquad \frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1, \quad \frac{(y-k)^2}{a^2} - \frac{(x-h)^2}{b^2} = 1.$$

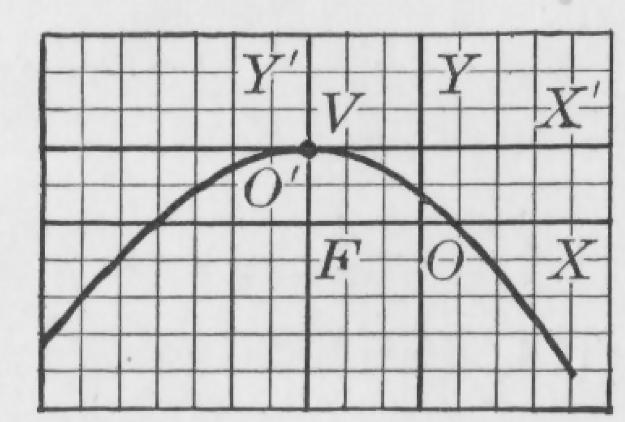
The center of the hyperbola is the point (h, k).

Example 1. — Plot the curve whose equation is

$$x^2 + 6x + 8y = 7.$$

Solution. — There is a squared term in x but not in y; we therefore try to write the equation in one of the forms (2). We have

$$x^2 + 6x = -8y + 7,$$
$$x^2 + 6x + 9 = -8y + 7 + 9,$$
$$(x + 3)^2 = -8(y - 2).$$



The curve is a parabola with vertex $(-3, 2)$. Translation of axes to this point as origin by the substitution

$$x = x' - 3, \quad y = y' + 2,$$

gives

$$x'^2 = -8y'.$$

The graph is readily drawn as shown in the figure. The directrix of the parabola is

$$y' = 2, \quad \text{or} \quad y = 4.$$

The coördinates of the focus with respect to the new axes are $(0, -2)$, and hence with respect to the old axes they are $(-3, 0)$.

Example 2. — Find an equation of the ellipse whose foci are the points $(2, -2)$, $(2, 6)$ and the length of whose minor axis is 6.

Solution. — The center is midway between the vertices, at $(2, 2)$; in equation (3), $h = 2$, $k = 2$. The distance between the foci is 8, hence $c = 4$. We have given $2b = 6$, that is, $b = 3$. Since $a^2 = b^2 + c^2$, we have $a = 5$. Hence the desired equation is

$$\frac{(x-2)^2}{9} + \frac{(y-2)^2}{25} = 1,$$

or

$$25x^2 + 9y^2 - 100x - 36y = 89.$$

EXERCISES

Draw the graph of each of the following equations, and find the coördinates of the vertices and foci, and the equations of the lines on which the axes lie, of the directrices, and of the asymptotes (for hyperbolas).

1. $y = x^2 - 4x + 4$.
2. $x^2 - 2x - 12y = 11$.
3. $y^2 - 12x + 12y = 12$.
4. $x^2 + 8x + 8y = 0$.
5. $x^2 + 4y^2 + 8x - 8y = 5$.
6. $4x^2 + 25y^2 - 8x + 50y = 35$.
7. $9x^2 + y^2 - 18x - 8y = 56$.
8. $16x^2 + y^2 + 16y = 105$.
9. $x^2 - y^2 + 20x = 0$.
10. $x^2 - 9y^2 + 4x + 18y = 30$.
11. $9x^2 - y^2 - 18x - 10y = 0$.
12. $16x^2 - y^2 + 32x - 20y + 60 = 0$.

all odd

Find for each of Exercises 13–24 the equation of the conic determined by the given conditions.

13. Parabola, vertex $(3, 4)$, axis on $y = 4$, latus rectum $= 8$.
14. Parabola, vertex $(4, -2)$, focus $(4, -6)$.
15. Parabola, focus $(-4, 0)$, directrix $y = 4$.
16. Parabola, ends of latus rectum $(-2, 2)$ and $(6, 2)$, vertex below latus rectum.

17. Ellipse, center $(-2, 2)$, vertex $(-2, 8)$, latus rectum $= 3$.
18. Ellipse, major axis $= 10$, foci $(4, -2)$ and $(6, -2)$.
19. Ellipse, minor axis $= 6$, focus $(4, 4)$, vertex $(4, 6)$.
★20. Ellipse, center $(-2, 4)$, focus $(2, 4)$, directrix $x = 14$.
21. Hyperbola, center $(2, -4)$, focus $(7, -4)$, vertex $(5, -4)$.
22. Hyperbola, foci $(-2, -6)$ and $(8, -6)$, conjugate axis $= 6$.
23. Hyperbola, center $(6, 4)$, vertex $(6, 8)$, conjugate axis $= 6$.
★24. Hyperbola, center $(2, 4)$, focus $(2, 14)$, eccentricity $= 2$.
25. Show that $y = Ax^2 + Bx + C$, where A, B, C are constants, $A \neq 0$, is the equation of a parabola. Find its vertex and focus.

66. Rotation of axes. Let the axes OX and OY be rotated through an angle θ to the positions OX' and OY', the origin remaining fixed. Let P be any point in the plane. Let its coördinates referred to the old axes be (x, y) and referred to the new axes be (x', y'). In Figure 74,

$$\text{angle } XOX' = \theta;$$
$$x = OR, \quad y = RP;$$
$$x' = OQ, \quad y' = QP.$$

We require the equations for rotating the axes, which express x and y in terms of x' and y'. We shall show that they are

$$(1) \qquad \begin{aligned} x &= x' \cos\theta - y' \sin\theta, \\ y &= x' \sin\theta + y' \cos\theta. \end{aligned}$$

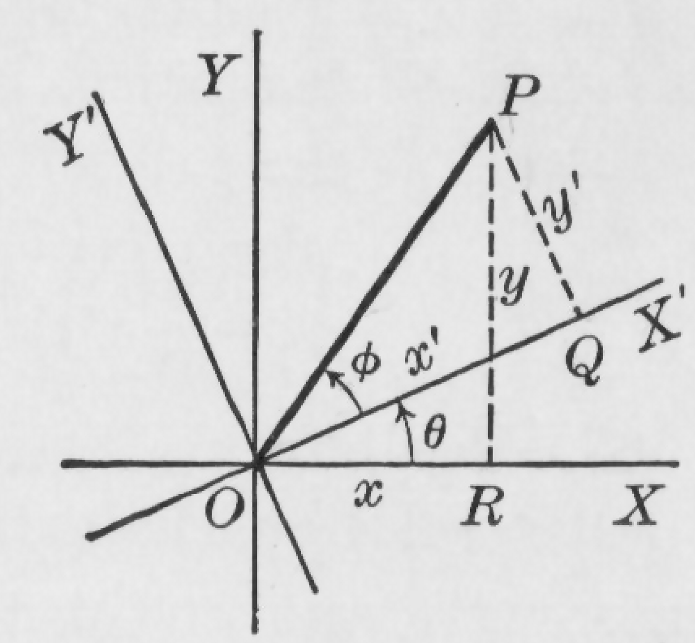


FIG. 74

Let the angle from OX' to OP be ϕ, and let $OP = r$.

Then

$$x' = r\cos\phi, \quad y' = r\sin\phi.$$

We now have

$$\begin{aligned} x &= r\cos(\phi + \theta) \\ &= r\cos\phi\cos\theta - r\sin\phi\sin\theta \\ &= x'\cos\theta - y'\sin\theta, \end{aligned}$$

which is the first of equations (1).

Similarly

$$\begin{aligned} y &= r \sin (\phi + \theta) \\ &= r \cos \phi \sin \theta + r \sin \phi \cos \theta \\ &= x' \sin \theta + y' \cos \theta. \end{aligned}$$

Thus we have the second of equations (1).

Solving (1) for x', y' in terms of x, y, we obtain

$$(2) \qquad \begin{aligned} x' &= x \cos \theta + y \sin \theta, \\ y' &= - x \sin \theta + y \cos \theta. \end{aligned}$$

It is to be noted that substitution (1) changes a term in x and y of a given degree m into a set of terms in x' and y' each of which is of the same degree m.

One of the most important applications of the rotation of axes is given in Chapter XII, § 108. It is there shown how by such a transformation an equation of the second degree in x and y which contains a term $2bxy$ can be reduced to an equation of the second degree in x' and y' which contains no $x'y'$ term. This is a step in the reduction of the general equation of the second degree

$$ax^2 + 2bxy + cy^2 + 2dx + 2ey + f = 0$$

to standard forms for which the graphs are known.

Example. — Find the coördinates of the point (4, 6) after the axes have been rotated through 30°.

Solution. — Since $\sin 30° = \frac{1}{2}$, $\cos 30° = \sqrt{3}/2$, we have

$$\begin{aligned} x' &= x \cos \theta + y \sin \theta = 2\sqrt{3} + 3, \\ y' &= - x \sin \theta + y \cos \theta = - 2 + 3\sqrt{3}. \end{aligned}$$

67. Equilateral hyperbola. An equilateral hyperbola is one whose transverse and conjugate axes are of equal length. If they lie on the coördinate axes the equation of the hyperbola is

$$(1) \qquad x^2 - y^2 = a^2.$$

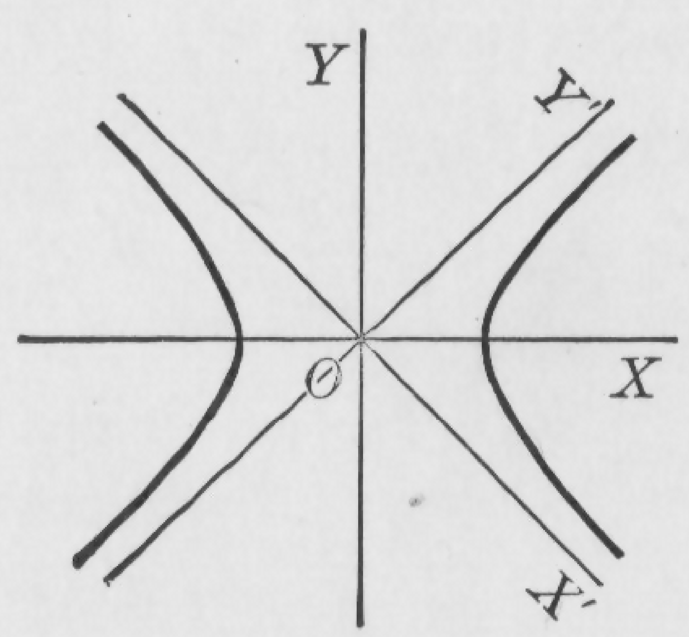


FIG. 75

The asymptotes, $x^2 - y^2 = 0$, or $y = x$ and $y = - x$, are mutually perpendicular, and pass through the origin. If the axes are rotated through $- 45°$ the asymptotes become the new axes. Since we have

$$\sin - 45° = - 1/\sqrt{2}, \cos - 45° = 1/\sqrt{2},$$

we obtain from (1), § 66,

$$x = \frac{x' + y'}{\sqrt{2}}, \qquad y = \frac{- x' + y'}{\sqrt{2}}.$$

Substituting in (1) and simplifying, we obtain

$$(2) \qquad \mathbf{2x'y' = a^2.}$$

This is the standard equation of an equilateral hyperbola referred to its asymptotes as axes.

It is to be noted that y' varies inversely as x' in (2); hence the graph which represents such variation is an equilateral hyperbola.

EXERCISES

Rotate the axes as indicated in each of the following Exercises 1–6, *and find the new coördinates of the given points. Draw a figure showing both pairs of axes and the points.*

1. $\theta = 90°$. $A(0, 6)$; $B(6, 0)$; $C(- 3, 3)$.

2. $\theta = 180°$. $A(0, 4)$; $B(4, 0)$; $C(- 6, 6)$.

3. $\theta = 60°$. $A(2, 2)$; $B(0, 4)$; $C(- 2, 0)$.

4. $\theta = - 30°$. $A(4, 0)$; $B(0, 4)$; $C(- 4, - 4)$.

5. $\theta = 120°$. $A(6, 0)$; $B(4, 4)$; $C(- 4, - 4)$.

6. $\theta = - 120°$. $A(6, 0)$; $B(4, 4)$; $C(- 4, - 4)$.

Rotate the axes as indicated in each of the following Exercises and transform the equation to correspond. Identify the curve and plot it.

7. $\theta = 45°$. $xy = 18$.

8. $\theta = \pi/4$. $xy = a^2/2$.

9. $\theta = 60°$. $x^2 + y^2 = a^2$.

10. $\theta = \pi/4$. $x^2 + y^2 - 4xy + 9 = 0$.

O **11.** $\theta = 45°$. $x^2 + y^2 + 2xy = 50$.

12. $\theta = -45°$. $x^2 + y^2 + 4xy = 25$.

13. $\theta = 90°$. $y^2 = 2px$.

O **14.** $\theta = 90°$. $b^2x^2 + a^2y^2 = a^2b^2$.

15. $\theta = -90°$. $b^2x^2 - a^2y^2 = a^2b^2$.

O **16.** $\theta = \tan^{-1}\frac{1}{3}$. $x^2 + 3xy - 3y^2 = 2$.

CHAPTER VIII

CERTAIN GENERAL METHODS

68. Two principal problems of analytic geometry. We have seen that by means of coördinates analytic geometry unites algebra, which deals with numbers and equations, and geometry, which deals with points and loci.

We recall the two important definitions:

a. An equation of a locus is an equation which is satisfied by the coördinates of each point on the locus, and is not satisfied by the coördinates of any point not on the locus.

b. The locus of an equation is the set of all points whose coördinates satisfy the equation.

We have been led to consider two principal problems:

I. *Given a geometrical locus, to find a corresponding equation.*

II. *Given an equation, to find the corresponding geometrical locus.*

These problems were stated in Chapter I, § 4, and the second was discussed in an elementary way. Both problems have been considered in the chapters on straight lines, circles, and conics. In the present chapter we develop certain methods of dealing with these problems which are applicable to more general loci and equations.

69. Finding the equation of a locus. To solve the first problem stated above, one may proceed by the following steps.

1. A figure is drawn showing the data for a representative point P on the locus. It is important that P be not chosen in any special position, but that it be a truly representative point.

2. If a coördinate system is not given, it must be introduced. The appropriate selection is often extremely important.

3. The description of the locus is written down as an equation of geometrical quantities involving P.

4. This equation is expressed in terms of the coördinates (x, y) of P, and is simplified algebraically.

5. It is shown that points not on the locus do not satisfy the simplified equation.

The simplified equation is the desired equation of the locus.

This method has been amply illustrated in the derivation of equations in the preceding chapters (for example, see pages 95, 111, 116, 123, and 125).

In the next chapter we shall show how a locus is described by means of a *pair of equations* containing a variable besides the coördinates of P.

EXERCISES

Find an equation of the locus of a point which moves as described below. When possible give the name of the curve.

1. Its distance from a given point A exceeds its distance from a given line BC by 4, the distance from A to BC being 10.

2. Its distance from a given line AB exceeds its distance from a given point C by 4, the distance from C to AB being 10.

3. The sum of the squares of its distances from two fixed points is a constant.

4. The difference of the squares of its distances from two fixed points is a constant.

5. The product of the squares of its distances from two fixed points is a constant. (Cassinian Oval.)

6. The ratio of the squares of its distances from two fixed points is a constant.

7. Its distance from the point $(0, 0)$ is half its distance from the line $y = 6$.

8. Its distance from the point $(0, 6)$ is a third of its distance from the line $x = -2$.

9. Its distance from the point $(0, 0)$ is twice its distance from the line $y = 6$.

10. Its distance from the point $(0, 6)$ is three times its distance from the line $x = -2$.

11. Its distance from the point $(0, 0)$ equals its distance from the line $x - y + 4 = 0$.

12. Its distance from the point $(2, 2)$ equals its distance from the line $x + y + \sqrt{8} = 0$.

13. Its distance from the point $(0, 0)$ is half its distance from the line $x - y + 4 = 0$.

14. Its distance from the point $(2, 2)$ is a third of its distance from the line $x + y + 4\sqrt{2} = 0$.

15. Its distance from the point $(0, 0)$ is twice its distance from the line $x - y + 4 = 0$.

16. Its distance from the point $(0, 0)$ is three times its distance from the line $x - y + 4 = 0$.

17. Its distance from the point $(0, 0)$ is e times its distance from the line $x = p$, where e and p are constants.

18. It forms with the points $A(2, 4)$ and $B(-2, 6)$ a triangle whose area is 10.

19. Its distance from the circle* $x^2 + y^2 = 4$ exceeds its distance from the line $x + y = 20$ by 3.

20. The sum of its distances from the circles* $(x - 4)^2 + y^2 = 4$ and $(x + 4)^2 + y^2 = 9$ is 5.

21. The difference of its distances from the circles* $(x - 4)^2 + y^2 = 4$ and $(x + 4)^2 + y^2 = 9$ is 3.

22. The product of its distances from two mutually perpendicular lines is a constant.

23. The product of its directed distances from lines $3x - 4y = 7$ and $3x + 4y = 2$ is 10.

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* By the distance of a point from a circle we mean the shortest distance from the point to the circle, directed so as to be positive if the point is outside of the circle, and negative if it is inside.

70. Discussion of the locus of an equation. The second problem, that of finding the locus of a given equation, is often solved by reducing the equation to a known form. This we have done in the case of linear equations and of certain equations of the second degree, the loci being straight lines, circles, or conics. In case this method fails, essential features of the locus can usually be determined, as we shall proceed to explain, by discussing it with regard to (a) *intercepts,* (b) *symmetry,* (c) *excluded value of coördinates,* and (d) *horizontal and vertical asymptotes.* In addition it may be desirable to find a few points by substituting values of one variable in the equation and calculating the corresponding values of the other.

We now take up in succession the four items (a), (b), (c), (d) in the discussion of the locus of an equation.

71. Intercepts. The **x-intercepts** of a curve are the abscissas of the points where the curve cuts the x-axis. The **y-intercepts** are the ordinates of the points where the curve cuts the y-axis.

Since $y = 0$ for all points on the x-axis, we find the x-intercepts of a curve by substituting $y = 0$ in its equation and solving for x. The y-intercepts are found similarly by setting $x = 0$ and solving for y.

Example. — Find the intercepts of the curve $y^2 = \dfrac{x^2 - 4}{x - 3}$.

Solution. — Setting $y = 0$ and solving for x, we find the x-intercepts to be ± 2. Thus the curve cuts the x-axis at $A(2, 0)$ and $B(-2, 0)$. Setting $x = 0$ and solving for y, we find the y-intercepts to be $\pm\sqrt{\frac{4}{3}}$. The curve cuts the y-axis at $C\left(0, \sqrt{\frac{4}{3}}\right)$ and $D\left(0, -\sqrt{\frac{4}{3}}\right)$.

72. Symmetry. We recall that two points are **symmetrical with respect to a given point** if the given point bisects the line joining the two points. Also that two points are **symmetrical with respect to a given line** when the given line

is the perpendicular bisector of the line segment joining the two points.

Thus, in Figure 76, R bisects PQ, hence P and Q are symmetrical with respect to the point R. Also AB is the perpendicular bisector of PQ; hence P and Q are symmetrical with respect to the line AB.

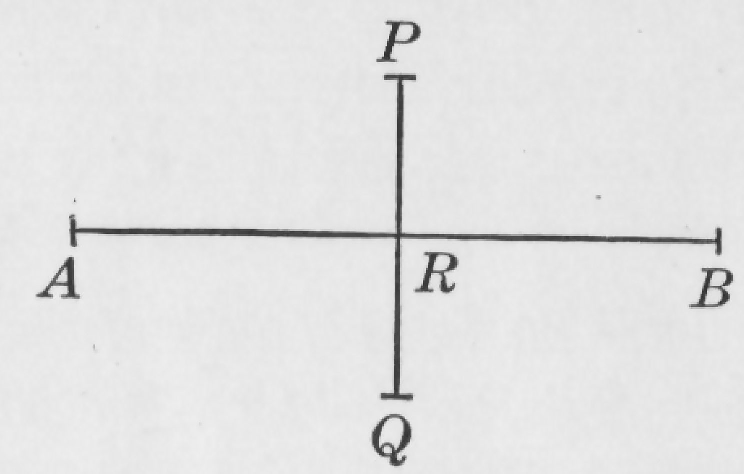

FIG. 76

If the points of a curve can be taken in pairs such that each pair is symmetrical with respect to the same given point R, then the **curve is symmetrical with respect to R,** and R is a **center of symmetry** of the curve.

Similarly, if the points of a curve can be paired in such a way that each pair is symmetrical with respect to the same given line, the curve is said to be **symmetrical with respect to the line,** and the line is an **axis of symmetry.**

How can we tell from the equation of a curve whether the curve is symmetrical with respect to the origin?

Let $P(x, y)$ be a point on the curve; the symmetrical point with respect to the origin is $P'(-x, -y)$. Hence *the curve is symmetrical with respect to the origin* if and only if the equation is satisfied by $(-x, -y)$ whenever it is satisfied by (x, y); that is, *if its equation reduces to the original form when* x *and* y *are replaced by* $-$ x *and* $-$ y.

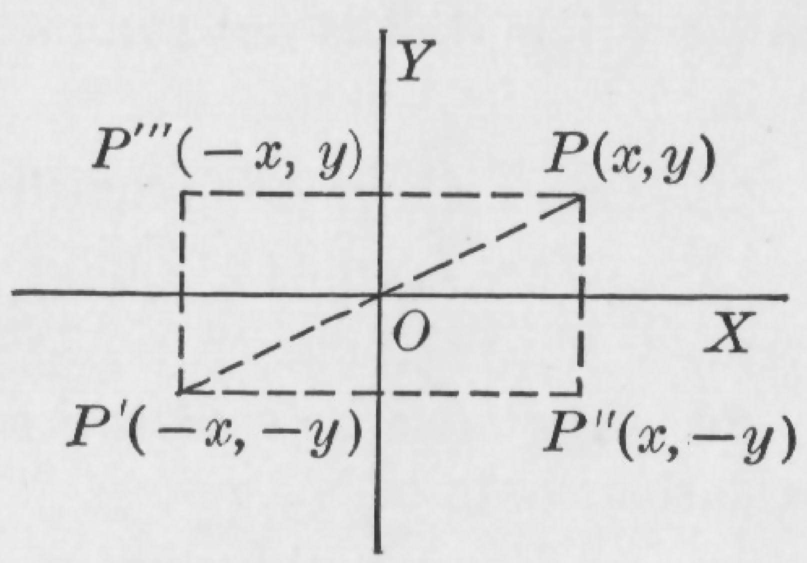

FIG. 77

Thus the ellipse $b^2x^2 + a^2y^2 = a^2b^2$ is symmetrical with respect to the origin since $b^2(-x)^2 + a^2(-y)^2 = a^2b^2$ reduces to the original form. Similarly the equilateral hyperbola $xy = 8$ is symmetrical with respect to the origin, since $(-x)(-y) = 8$ reduces to the same equation.

The symmetrical point to $P(x, y)$ with respect to the

x-axis is $P''(x, -y)$. Hence *the curve is symmetrical with respect to the* x-*axis if and only if its equation reduces to the original form when* y *is replaced by* $-$ y.

Similarly *the curve is symmetrical with respect to the* y-*axis if and only if its equation reduces to the original form when* x *is replaced* by $-$ x.

EXERCISES

For the locus of each of the following equations find the x- *and the* y-*intercepts, and test for symmetry with respect to the origin and the coördinate axes.*

1. $x + y = 0$. **2.** $3x + y = 6$.

3. $x^2 + y^2 = 25$. **4.** $4x^2 + 9y^2 = 36$.

5. $9x^2 - 4y^2 = 36$. **6.** $x^2 + y^2 + 4x = 12$.

7. $y^2 = 8x$. **8.** $x^2 = 16y$.

9. $y^2 = x^2 + x$. **10.** $y = x^3$.

11. $y^2 = x^3$. **12.** $(x^2 + y^2)^2 = a^2(x^2 - y^2)$.

13. $x^2y = x + y$. **14.** $y^2 = \dfrac{x^2 - 4}{x - 3}$.

15. $y^3 = x^3 - x$. **16.** $x^3 + y^3 = x - y$.

17. $x^2 + xy + y^2 = 7$.

18. Give an example of a curve which is symmetrical with respect to the origin, but is not symmetrical with respect to either coördinate axis.

19. Can a curve be symmetrical with respect to (*a*) both axes but not the origin ? (*b*) the origin and one axis but not the other ? (*c*) one axis but not the other or the origin?

20. What can be said concerning the symmetry of a curve whose equation contains

(*a*) only terms of even degree in x?
(*b*) only terms of even degree in y?
(*c*) only terms of even degree in x and y?
(*d*) only terms of odd degree in x and y?

Note. To illustrate what is here meant we remark that the terms x^4, y^6, xy are all of even degree in x and y, but xy is of odd degree in x and of odd degree in y. See footnote, page 2.

73. Excluded values of coördinates. It may happen that corresponding to some values of one variable, say x, there is no real value of y. Since both coördinates of a point are real, there are no points on the curve having these abscissas. We say that they are **excluded values** of x.

If it turns out that all values of x between two values, a and b, are excluded, it follows that there is no portion of the curve between the lines $x = a$ and $x = b$.

The following examples illustrate how we find excluded values of the coördinates.

Example 1. — Find excluded values of x and y for the equation

$$9x^2 - 4y^2 = 36.$$

Solution. — Solving for y, we have

$$y^2 = \frac{9x^2 - 36}{4} = \frac{9(x^2 - 4)}{4},$$

$$y = \pm \frac{3}{2}\sqrt{x^2 - 4}.$$

The values for y are imaginary if and only if the expression under the radical is negative. Hence the values of x which are excluded are those which make $x^2 < 4$; that is, values of x between -2 and 2. The curve does not extend into the region between the lines $x = -2$ and $x = 2$.

Solving for x we have

$$x = \pm \frac{2}{3}\sqrt{9 + y^2}.$$

Hence x is imaginary if and only if $9 + y^2$ is negative. But $9 + y^2$ is always positive; thus no values of y are excluded, and the curve extends indefinitely far.

The preceding conclusions concerning the extent of the curve are checked by constructing the curve, which is a hyperbola, by the methods of Chapter VI, § 52, page 127.

Example 2. — Find excluded values of x and y for the equation

$$x^2 + y^2 + 6x - 16 = 0.$$

Solution. — Solving for y, we have

$$y = \pm\sqrt{16 - 6x - x^2} = \pm\sqrt{(8 + x)(2 - x)}.$$

The value of y is imaginary if and only if

$$(8 + x)(2 - x) < 0.$$

In the adjoining figure we represent diagrammatically the facts concerning the signs of the two factors. The product will be negative if one factor is +, the other −. This occurs for $x > 2$ and for $x < -8$; such

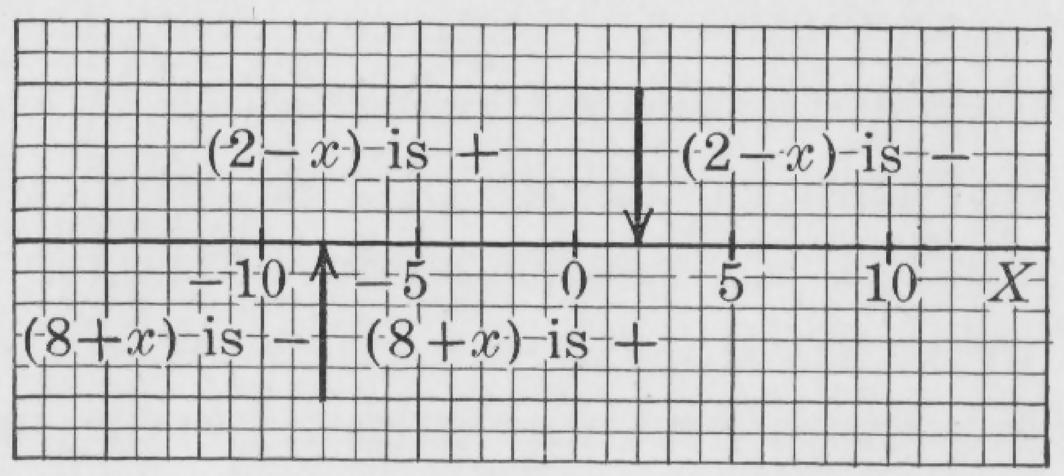

Fig. 78

values of x are excluded. The curve does not extend to the right of the line $x = 2$, nor to the left of the line $x = -8$.

On solving for x, we obtain

$$x = \frac{-6 \pm \sqrt{36 - 4(y^2 - 16)}}{2} = -3 \pm \sqrt{25 - y^2}.$$

Then x is imaginary if and only if $25 - y^2 < 0$; that is, if $y > 5$ or $y < -5$. The curve does not extend above the line $y = 5$, nor below the line $y = -5$.

The curve therefore lies in the rectangle whose sides are on the lines

$$x = 2, \quad x = -8, \quad y = 5, \quad y = -5.$$

The curve may be recognized as a circle (Chapter V) and the statements concerning extent are readily verified.

Example 3. — Find excluded values of x and y for the equation

$$x^2 - xy - 2x + 3y = 0.$$

Solution. — Solving for y, we have

$$y = \frac{x^2 - 2x}{x - 3}.$$

No value of x gives an imaginary value of y. But if we take $x = 3$ there is a division by zero, which is impossible. Hence $x = 3$ is excluded.

The solution for x is

$$x = \frac{(y + 2) \pm \sqrt{(y + 2)^2 - 12y}}{2} = \frac{(y + 2) \pm \sqrt{R}}{2},$$

where

$$R = y^2 - 8y + 4.$$

If we solve the quadratic equation $R = 0$, we find two roots,

$$y_1 = 4 - 2\sqrt{3}, \qquad y_2 = 4 + 2\sqrt{3};$$

hence R may be written

$$R = (y - y_1)(y - y_2).$$

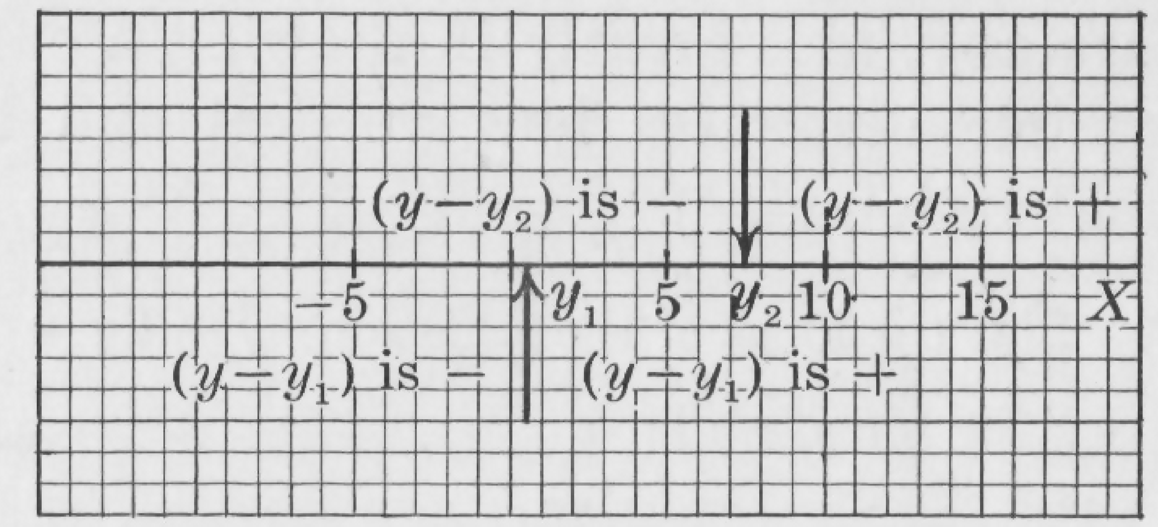

Fig. 79

By an argument similar to that of Example 2, it follows that R is negative for values of y between y_1 and y_2, and hence x is imaginary for such values. Thus values of y between $4 - 2\sqrt{3}$ and $4 + 2\sqrt{3}$ are excluded; there is no portion of the locus of the equation between the lines

$$y = 4 - 2\sqrt{3}, \qquad y = 4 + 2\sqrt{3}.$$

EXERCISES

Find the extent of each of the curves having the following equations, using one of the methods given in the Examples of § 73.

1. $x + y = 0.$ **2.** $ax + by + c = 0.$

3. $y^2 = 2px.$ **4.** $x^2 = -2py.$

5. $x^2 + y^2 - 8x = 9.$ **6.** $x^2 + y^2 + 4x - 6y = 12.$

7. $4x^2 + 9y^2 = 144.$ **8.** $9x^2 + 25y^2 = 225.$

9. $25x^2 - 4y^2 = 100.$ **10.** $4x^2 - 25y^2 = -100.$

11. $4x^2 - y^2 + 8x + 6y = 41.$ **12.** $9x^2 + 25y^2 - 18x + 200y = -184.$

13. $x^2 - 2xy + y^2 - 5x = 0.$ **14.** $4xy + 4y^2 - 2x + 3 = 0.$

15. $x^2 + xy + y^2 + 3y = 0.$ **16.** $x^2 - 2xy + 4y^2 - 4x = 0.$

17. $3x^2 + 4xy + y^2 - 2x - 1 = 0.$ **18.** $5x^2 + 4xy - y^2 + 24x - 6y = 5.$

19. $(x^2 + y^2)^2 = 16(x^2 - y^2).$ **20.** $y^2 = 2x(x - 5)(x + 6).$

21. $y^2 = \dfrac{x - 5}{(x + 2)(x - 1)}.$ **22.** $x^2 = \dfrac{y^2 - 9}{y^2 - 25}.$

74. Horizontal and vertical asymptotes. If a point moving continuously along a curve recedes indefinitely far from the origin and at the same time approaches indefinitely near to a given straight line, the line is called an **asymptote** of the curve (compare page 130). The location of asymptotes, when there are any, aids greatly in drawing a curve. The following example illustrates how vertical and horizontal asymptotes may be found.

Example 1. — Plot the locus of the equation $xy - 2y = 12$.

Solution. — Solving for y, we have

$$y = \frac{12}{x - 2}.$$

We substitute values of x and calculate y; results are given in the following table.

x	-4	-2	-1	0	1	2	3	4	5	6	8
y	-2	-3	-4	-6	-12	—	12	6	4	3	2

The value $x = 2$ leads to division by zero, which is impossible; hence there is no point on the curve at which $x = 2$. If we take values of x very near to 2 we get values of y as follows:

x	1.7	1.8	1.9	1.98	1.99	2.2	2.1	2.02	2.01
y	-40	-60	-120	-600	-1200	60	120	600	1200

Thus as x approaches indefinitely near to 2, passing through values which are always less than 2, the values of y are negative and increase numerically without limit. Hence the curve goes down indefinitely far, approaching the line $x = 2$ from the left; this line is a vertical asymptote. As x approaches 2 passing through values which are always greater than 2, the values of y are positive and increase indefinitely. Hence the line $x = 2$ is a vertical asymptote of a branch of the curve which rises as we approach $x = 2$ from the right.

If we solve the original equation for x we have

$$x = 2 + \frac{12}{y}.$$

We are led to division by 0 if we substitute $y = 0$, hence there is no point of the curve on the x-axis. Substitution of a series of values of y less than zero and approaching that value gives a series of values of x which increase numerically but are negative. Thus the x-axis is a horizontal asymptote of a branch of the curve which goes out to the left beneath it.

Similarly if we take a series of positive values of y which approach indefinitely near to zero, we obtain a series of positive values of x which increase indefinitely. Hence the x-axis is an asymptote of a branch of the curve which goes out to the right above the asymptote.

The graph is shown in Figure 80.

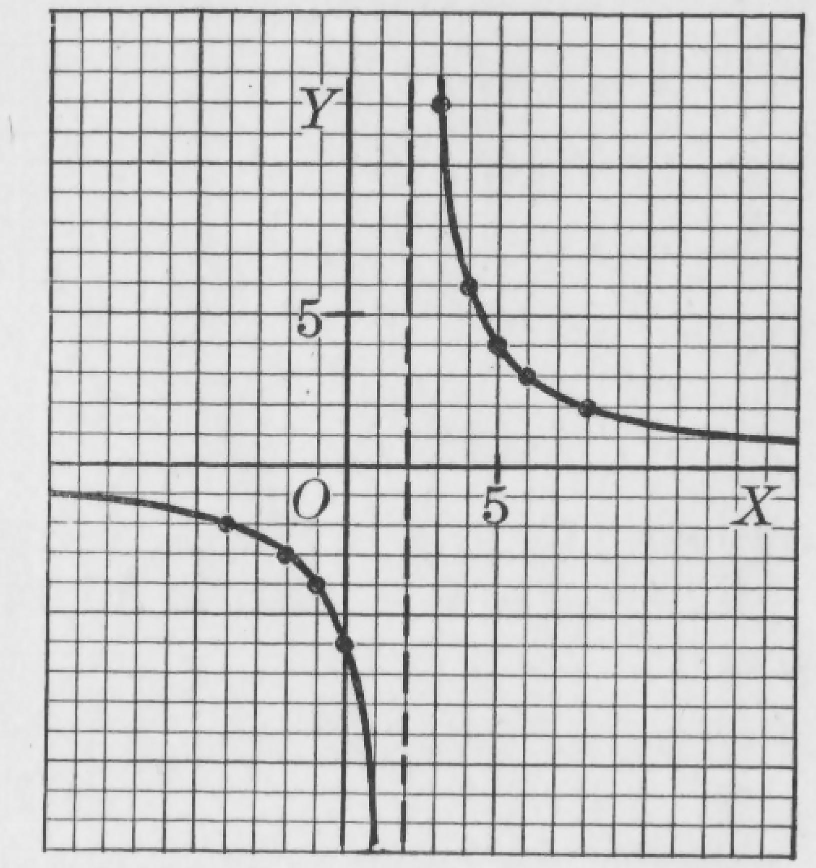

FIG. 80

It is clear from the example that we may find asymptotes of the locus of an algebraic equation as follows.

Solve the equation for y and simplify. Find values of x, if any, which make a denominator of the expression for y vanish. If $x = a$ is such a value, and the curve approaches the line $x = a$ indefinitely, then the line $x = a$ is a vertical asymptote. Examine the behavior of y for values of x approaching a from each side to determine how the curve approaches its asymptote.

Next solve the equation for x, and proceed similarly to find horizontal asymptotes.

Example 2. — Discuss the locus of the equation

$$y = \frac{x^2}{x^2 - 25}$$

in the vicinity of its vertical asymptotes.

Solution. — We have two vertical asymptotes,

$$x = 5, \quad x = -5.$$

When x approaches -5 from the left, through such values as $x = -6$, -5.5, -5.1, $\cdot\cdot\cdot$, we find that y is positive and increasing. The

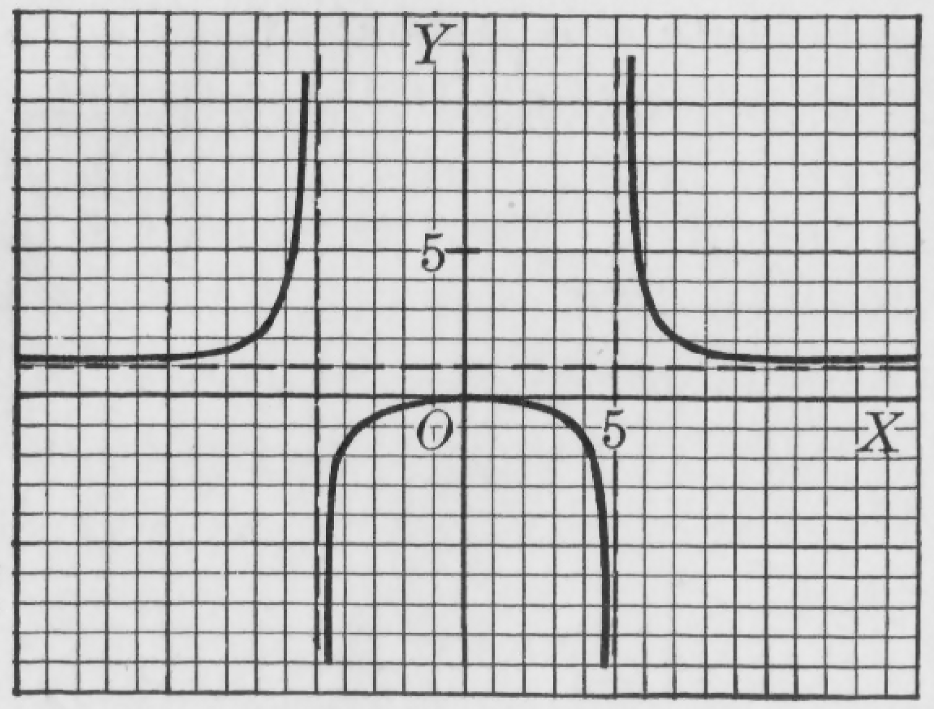

FIG. 81

curve rises to the left of the asymptote $x = -5$. When x approaches -5 from the right, that is, through such values as $x = -4, -4.5, -4.9, \cdots$, we find that y is negative and increasing numerically. Hence the curve goes down as it approaches this asymptote from the right.

A similar discussion shows that the curve goes down to the left of the asymptote $x = 5$, and goes up as it approaches $x = 5$ from the right.

The curve is shown in Figure 81. Note the horizontal asymptote $y = 1$.

EXERCISES

Discuss the locus of each of the following equations for (a) *intercepts,* (b) *symmetry,* (c) *excluded values of coördinates,* (d) *horizontal and vertical asymptotes; and plot the curve.*

1. $xy + x + 8 = 0$. **2.** $x^2 + y = 8$.

3. $x^2 + xy = 12$. **4.** $xy - y^2 = 6$.

5. $x^2 + 2xy - y^2 = 18$. **6.** $x^2 - 2xy + y = 4$.

7. $x^2y = 12$. **8.** $xy^2 - 9x = 6$.

9. $(x - y)(x + y) = 12x$. **10.** $x^4 - x^2 + y^2 = 0$.

11. $x^2y - 4x^2 = 4y$. **12.** $y^2 = 2x(x - 5)(x + 6)$.

sat.

13. $y = \dfrac{x-5}{(x+2)(x-1)}$. **14.** $y^2 = \dfrac{x-5}{(x+2)(x-1)}$.

15. $y = \dfrac{x}{x^2-4}$. **16.** $y^2 = \dfrac{x}{x^2-4}$.

17. $y = \dfrac{x^2-4}{x^2+4}$. **18.** $y^2 = \dfrac{x^2-4}{x^2+4}$.

separate sheet

19. $y = \dfrac{x^2-4}{x^2-3x-4}$. **20.** $y^2 = \dfrac{x^2-4}{x^2-3x-4}$.

21. $y = \dfrac{(x-2)^2}{x^2}$. **22.** $y^2 = \dfrac{(x-2)^2}{x^2}$.

23. $y = \dfrac{12}{(x^2-4)(x^2-36)}$. **24.** $y^2 = \dfrac{12}{(x^2-4)(x^2-36)}$.

next time

75. Factorable equations. Consider the equation

$$(1) \qquad (x + y - 2)(x + 2y - 3) = 0.$$

This is satisfied by all points which make

$$(2) \qquad x + y - 2 = 0,$$

also by those which make

$$(3) \qquad x + 2y - 3 = 0,$$

but by no other points. Hence the locus of equation (1) consists of the two straight lines whose equations are (2) and (3).

In general the locus of an equation whose left member is factorable, the right member being zero, consists of the loci of the equations obtained by setting each factor equal to zero. Thus the locus of

$$f_1(x, y)f_2(x, y) = 0,$$

where $f_1(x, y)$ and $f_2(x, y)$ represent factors containing the letters x and y, consists of the two loci

$$f_1(x, y) = 0, \quad f_2(x, y) = 0.$$

EXERCISES

Plot the loci of each of the following equations.

1. $(x + y - 7)(x^2 + y^2 - 25) = 0$.

2. $x^2 - 2xy - 3y^2 = 0$.

3. $(x + y)^2 - 5(x + y) + 4 = 0$.

4. $(x + 2y)^2 + 6 = 5(x + 2y)$.

5. $(x^2 + y^2)^2 + 100 = 29(x^2 + y^2)$.

6. $(x^2 + 4y^2)^2 - 25(x^2 + 4y^2) = 0$.

7. $(x + 2)(3x - y)(x + 3y) = 0$.

8. $x^2y^2 - 36 = 0$.

9. $x^2 - y^2 + x - y = 0$.

10. $x^3 + y^3 + x + y = 0$.

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★ 76. Intersections of a curve and a straight line. Suppose we have given the equation of a curve,

$$(1) \qquad f(x, y) = 0,$$

where $f(x, y)$ represents an expression containing x and y; and let

$$(2) \qquad l(x, y) = 0$$

be the equation of a straight line, where $l(x, y)$ represents an expression of the first degree in x and y.

In order that we may discuss the intersections of the loci of equations (1) and (2) we assume that $l(x, y)$ is not a factor of $f(x, y)$; if $l(x, y)$ were a factor of $f(x, y)$ then the locus of (1) would contain the line (2).

As was shown in Chapter I, § 5, page 21, the coördinates of each point of intersection will satisfy both equations; to find these coördinates we solve the pair of equations (1) and (2), each solution giving a point of intersection.

Suppose equation (1) is of the third degree in x and y, and that (2) can be solved for y. From (2) it follows that y is equal to an expression of not more than the first degree in x. When this is substituted in (1) we therefore get an equation of third degree or less in x. It is a theorem of algebra that such an equation has at most three roots. Hence we obtain at most three values of x in solving (1) and (2). Corresponding to each of these there is found just one value of y from equation (2), and thus the cubic curve (1) and the straight line (2) intersect in at most three points.

By an extension of the preceding reasoning we obtain the following theorem:

The straight line (2) *intersects the curve whose equation* (1) *is of the* n*th degree in at most* n *points.*

Since every conic has an equation of the second degree it follows that *a conic can be cut by a straight line in at most two points.*

★ 77. Intersections of curves. Let the equations of two curves be

$$(1) \qquad f(x, y) = 0, \quad g(x, y) = 0,$$

where $f(x, y)$ and $g(x, y)$ are two expressions in x and y. The coördinates of a point of intersection satisfy both equations. Hence the points of intersection are found by solving them as simultaneous equations.

In solving we usually derive new equations from the given ones. In particular we may, in order to eliminate certain terms, multiply each equation by a constant and add, obtaining the equation

$$(2) \qquad af(x, y) + bg(x, y) = 0,$$

where a and b are constants. We ask now, what can be said of the geometrical relations among the curves whose equations are given in (1) and (2)?

It is not difficult to see that no matter what values are given to a and b, the curve (2) will pass through the points of intersection of the curves (1); for if $P_1(x_1, y_1)$ is a point of intersection we must have $f(x_1, y_1) = 0$ and $g(x_1, y_1) = 0$, and hence when we substitute x_1, y_1 in (2) the equation is satisfied.

Furthermore the curve (2) has no point of intersection with either of the curves (1) which is not on both of them, if a and b are both different from zero. To show this, we let $P_1(x_1, y_1)$ be a point on the curve (2) and the curve $g(x, y) = 0$, and prove that it lies on the curve $f(x, y) = 0$. We may write

$$f(x, y) \equiv \frac{1}{a}[af(x, y) + bg(x, y)] - \frac{b}{a}g(x, y).$$

Since the right member vanishes when we substitute (x_1, y_1), we have $f(x_1, y_1) = 0$, which was to be proved.

Example. — Find the points of intersection of the curves

$$(1) \qquad x^2 + 4y^2 - 25 = 0,$$

$$(2) \qquad 4x^2 - 8y^2 - 25 = 0,$$

and give the geometrical representation of equations used in finding the points.

Solution. — The first equation is that of an ellipse, the second that of a hyperbola (Fig. 82). To solve for x and y by one method, we multiply the first by -1, the second by 1 and add; we obtain

$$(3) \qquad 3x^2 - 12y^2 = 0.$$

This equation factors, and may be written

$$3(x - 2y)(x + 2y) = 0.$$

It is the equation of a pair of straight lines passing through the intersections of (1) and (2), namely, the lines

$$(4) \qquad x - 2y = 0,$$
$$(5) \qquad x + 2y = 0.$$

To find the intersections of the first line with the curve (1), substitute $x = 2y$ in (1); we get

$$(6) \qquad 8y^2 - 25 = 0 \quad \text{or} \quad y = \pm\frac{5}{4}\sqrt{2}.$$

This equation may be regarded as the equation of two lines parallel to the x-axis and passing through the required points. If the above values of y are substituted in (4) we next have

$$(7) \qquad x = \pm\frac{5\sqrt{2}}{2},$$

which represents two lines parallel to the y-axis and passing through the required points of intersection. Hence (4) cuts (1) in the points

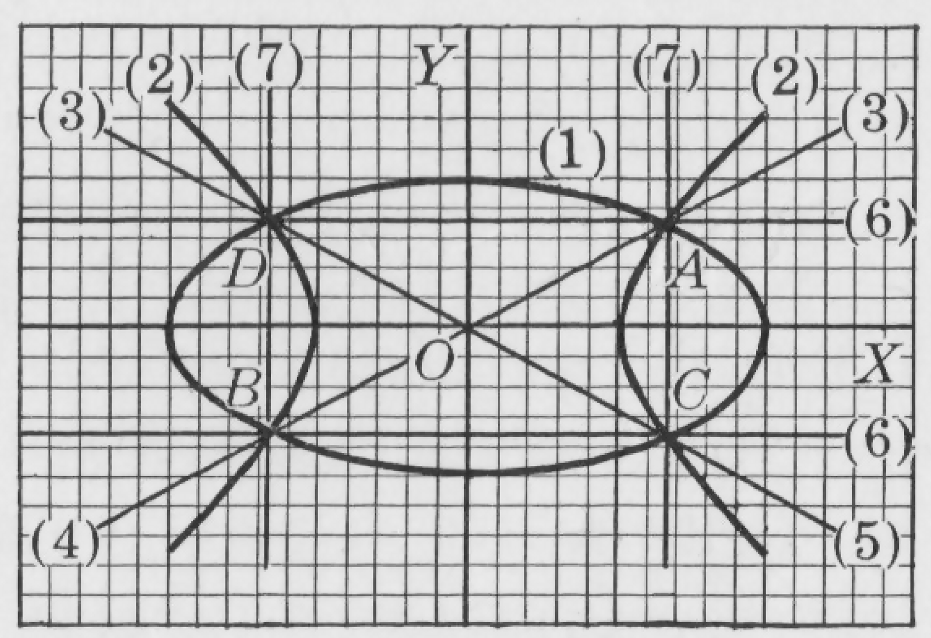


FIG. 82

$A\left(\frac{5\sqrt{2}}{2}, \frac{5\sqrt{2}}{4}\right)$ and $B\left(\frac{-5\sqrt{2}}{2}, \frac{-5\sqrt{2}}{4}\right)$. The student may show that the line (5) cuts (1) in the points $C\left(\frac{5\sqrt{2}}{2}, \frac{-5\sqrt{2}}{4}\right)$ and $D\left(\frac{-5\sqrt{2}}{2}, \frac{5\sqrt{2}}{4}\right)$.

Thus the curves (1) and (2) intersect at A, B, C, and D.

EXERCISES

Find the points of intersection of the curves whose equations are as follows.

1. $x + 2y = 5,$ $2x - y = 5.$
2. $2x + 3y = 8,$ $3x - y = 1.$
3. $x^2 + y^2 - 4x + 6y = 12,$ $x + y + 1 = 0.$
4. $x^2 + y^2 + 8x - 2y = 8,$ $x - 2y + 4 = 0.$
5. $x^2 + 4y^2 = 100,$ $x + 2y = 2.$
6. $x^2 - 4y^2 = 33,$ $2x - 5y = 4.$
7. $x^2 + y^2 = 16,$ $y^2 = 6x.$
8. $x^2 - y^2 = 16,$ $x^2 = 8y.$

don't graph

Find algebraically the points of intersection of the following curves. Plot the curve for each of these equations and for each equation used in obtaining the solutions.

9. $x^2 + xy + y^2 = 28,$ $x^2 - xy + y^2 = 12.$
10. $x^2 + xy + y^2 = 19,$ $xy = 6.$
11. $x^2 + 4xy + y^2 = 61,$ $x^2 - 4xy + y^2 = 13.$
12. $x^2 + y^2 + (x - y)^2 = 42,$ $x^2 + y^2 = 41.$
13. $x^2 + y^2 = 25,$ $x^2 + y^2 + (x - y)^2 = 25.$
14. $x^3 + y^3 = 3xy,$ $x - 5y + 6 = 0.$

Hint. In Exercise 14 let $y = tx$, and find t; then find y and x.

★ 78. Trigonometric curves. We have considered heretofore only algebraic equations. A few curves which are loci of trigonometric equations will now be studied.

(*a*) *The Sine Curve,* y = sin x. In advanced mathematics it is desirable to use the *radian* as the unit of angular measure, and this we shall do here. We recall * that $\pi = 3.14 \cdots$, and that

$$\pi \text{ radians} = 180°,$$

$$1 \text{ radian} = \frac{1}{\pi} \cdot 180° = 57.3°.$$

* See page 6.

For the equation

$$y = \sin x$$

we have the following table (see page 11).

x radians	0	$\frac{\pi}{6} = .52$	$\frac{\pi}{3} = 1.05$	$\frac{\pi}{2} = 1.57$
$y = \sin x$	0	.50	.87	1.00

By use of the formulas (see page 8)

$$\sin(\pi - x) = \sin x, \qquad \sin(\pi + x) = -\sin x,$$
$$\sin(2\pi + x) = \sin x, \qquad \sin -x = -\sin x,$$

the table may be extended indefinitely. For example we have

$$\sin\frac{2\pi}{3} = \sin\left(\pi - \frac{\pi}{3}\right) = \sin\frac{\pi}{3} = .87;$$

$$\sin\frac{5\pi}{6} = \sin\left(\pi - \frac{\pi}{6}\right) = \sin\frac{\pi}{6} = .50;$$

$$\sin\frac{7\pi}{6} = \sin\left(\pi + \frac{\pi}{6}\right) = -\sin\frac{\pi}{6} = -.50.$$

As an extension of the preceding table we have the following:

x radians	$\frac{2\pi}{3} = 2.09$	$\frac{5\pi}{6} = 2.62$	$\pi = 3.14$	$\frac{7\pi}{6} = 3.67$	$\frac{4\pi}{3} = 4.19$	$\frac{3\pi}{2} = 4.71$	$\frac{5\pi}{3} = 5.24$
$y = \sin x$	.87	.50	0	$-.50$	$-.87$	-1.00	$-.87$

The student should extend the table further. The resulting curve is indicated in Figure 83; it is called the *sine curve.** It oscillates from $y = -1$ to $y = 1$, in a periodic wave, the period being 2π.

* The *cosine curve*, $y = \cos x$, differs from the sine curve very little; it can be obtained from the latter by merely shifting the sine curve to the left through the distance $\pi/2$.

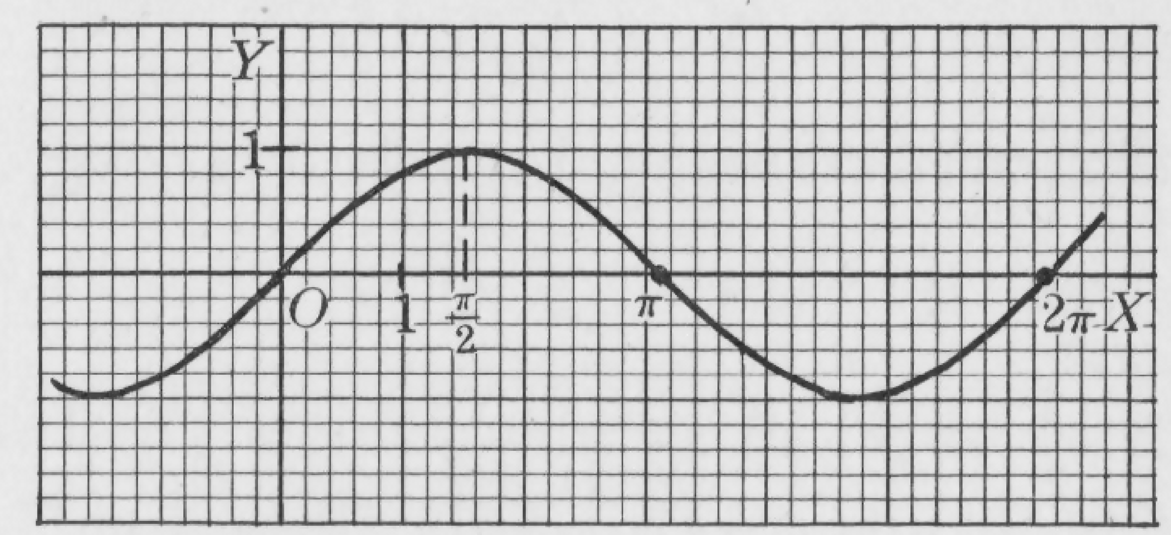


FIG. 83. *Sine Curve*

(*b*) *The curve* y = a sin kx.

Consider the curve whose equation is

$$y = 2 \sin 3x.$$

A graph is obtained from the preceding by giving x a series of values such that $3x$ takes on the values 0, $\pi/6$, $\pi/3$, $\pi/2$, $\cdots$. The values of y are then twice those given in the preceding tables. We obtain the following table.

x	0	.17	.35	.52	.70	.87	1.05	1.22	1.40	1.57	1.75
y	0	1.0	1.74	2.0	1.74	1.0	0	-1.0	-1.74	-2.0	-1.74

The values repeat when $3x$ is increased by 2π, or, in other words, when x is increased by $2\pi/3$; thus the curve oscillates with a period of $2\pi/3$. The values of y vary from -2 to 2; that is, the *amplitude* of the oscillation is 2. The graph is shown in Figure 84.

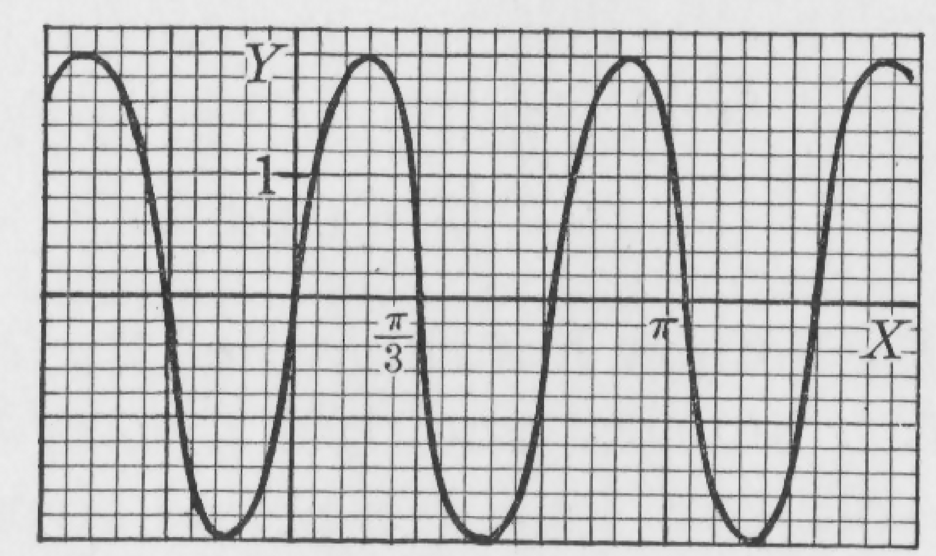


FIG. 84

Similarly the curve whose equation is

$$y = a \sin kx$$

has the form of a wave, oscillating from $y = -a$ to $y = a$, having an *amplitude* a and a *period* $2\pi/k$.

(*c*) *The curve* y = a sin kx + b sin lx.

Consider the curve whose equation is

$$y = \sin x - 2 \sin 3x.$$

This curve is obtained by combination of the two preceding curves, subtracting for each value of x the ordinate of $y = 2 \sin 3x$ from the ordinate of $y = \sin x$. The resulting curve is shown in Figure 85. It is periodic with the period 2π.

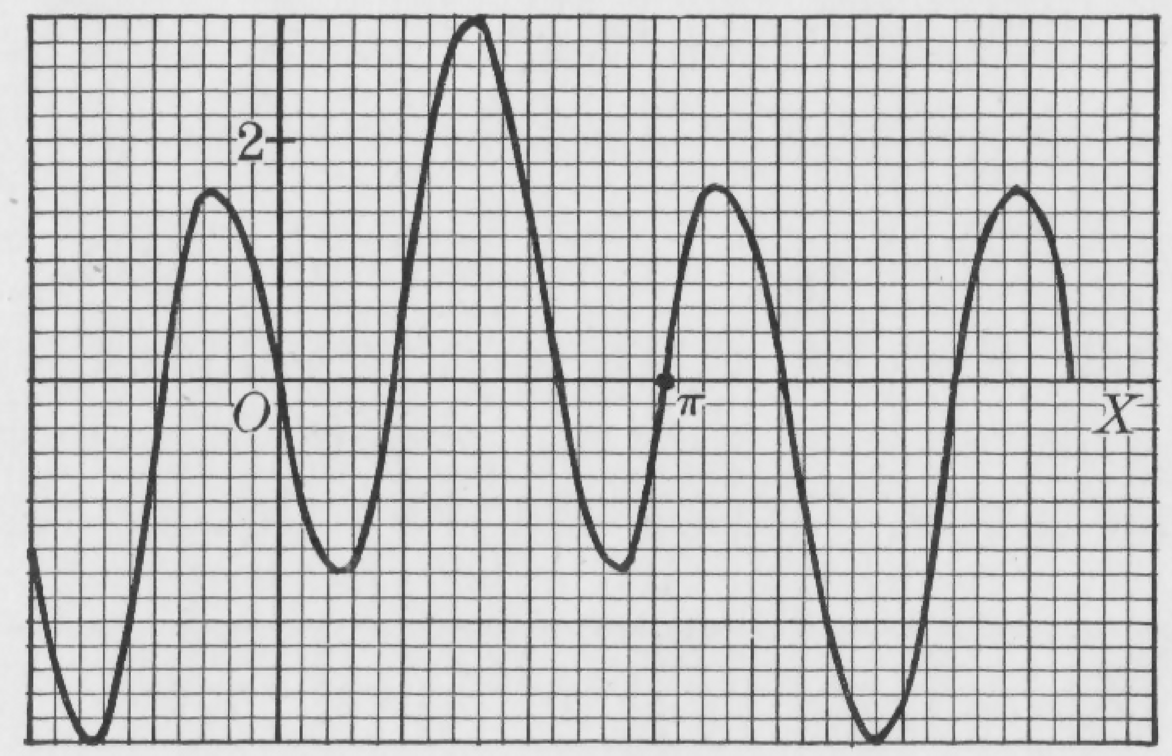


Fig. 85

Similarly the locus of the equation

$$y = a \sin kx + b \sin lx$$

can be obtained by adding ordinates of the two curves

$$y = a \sin kx, \quad y = b \sin lx.$$

The former has the period $2\pi/k$, the latter the period $2\pi/l$. Any integral multiple of $2\pi/k$ which is also an integral multiple of $2\pi/l$ is a period of the resulting curve.

It can be shown that any sufficiently smooth curve which has a period 2π can be represented by an equation of the form

$$y = a_0 + a_1 \cos x + a_2 \cos 2x + a_3 \cos 3x + \cdots$$
$$+ b_1 \sin x + b_2 \sin 2x + b_3 \sin 3x + \cdots.$$

EXERCISES

Draw the graph of each of the following equations.

1. $y = \cos x$. **2.** $y = 3 \cos 2x$.

3. $y = \cos x + 3 \cos 2x$. **4.** $y = \sin x + \cos x$.

5. $y = \sin x + \frac{1}{2} \sin 2x + \frac{1}{3} \sin 3x$.

6. $y = \sin x - \sin 2x + \sin 3x - \sin 4x$.

7. $y = x + \sin \pi x$. **8.** $y = \frac{1}{4}x^2 - \cos \pi x$.

9. $y = x \sin \pi x$. **10.** $y = \sin \dfrac{\pi}{x}$.

11. $y = \sin^{-1} x$.

Hint. This is equivalent to the equation $x = \sin y$.

12. $y = \frac{1}{2} \sin^{-1} 3x$. **13.** $y = \tan x$.

14. $y = \tan^{-1} x$. **15.** $y = \sec x$.

★ **79. Exponential and logarithmic curves.** A curve whose equation is of the form

$$(1) \qquad y = ae^{bx}$$

where a, e, and b are constants, is called an **exponential** curve.

It develops in the branch of higher mathematics known as the *Calculus* that equations of the form (1) are simplest when the constant e has a particular value, namely

$$e = 2.71828 \cdots.$$

This number is called the *natural base* of logarithms.

Consider the special case

$$y = e^x.$$

Extensive tables of values of the *exponential function* e^x have been made, from which we have formed the small table which follows.

Table of values of the exponential function e^x

x	.0	.1	.2	.3	.4	.5	.6	.7	.8	.9
0	1.00	1.11	1.22	1.35	1.49	1.65	1.82	2.01	2.23	2.46
1	2.72	3.00	3.32	3.67	4.06	4.48	4.95	5.47	6.05	6.69
2	7.39	8.17	9.03	9.97	11.0	12.2	13.5	14.9	16.4	18.2
3	20.1	22.2	24.5	27.1	30.0	33.1	36.6	40.4	44.7	49.4
4	54.6	60.3	66.7	73.7	81.5	90.0	99.5	110.	122.	134.
5	148.	164.	181.	200.	221.	245.	270.	299.	330.	365.
− 0	1.00	0.90	0.82	0.74	0.67	0.61	0.55	0.50	0.45	0.41
− 1	0.37	.33	.30	.27	.25	.22	.20	.18	.17	.15
− 2	.14	.12	.11	.10	.09	.08	.07	.07	.06	.06
− 3	.05	.05	.04	.04	.03	.03	.03	.02	.02	.02
− 4	.02	.02	.01	.01	.01	.01	.01	.01	.01	.01
− 5	.01	.01	.01	.00	.00	.00	.00	.00	.00	.00

To find the value, for example, of $e^{2.7}$ we look in the x column for 2, then follow across the row to the .7 column, finding the desired value, 14.9. Similarly we find that $e^{-2.7} = 0.07$.

By use of the table we plot the curve shown in Figure 86. Since, for any number x,

$$e^{x+1} = e \cdot e^x = 2.72e^x,$$

it follows that when x is increased indefinitely e^x increases indefinitely. The curve rises steeply in the first quadrant. Since

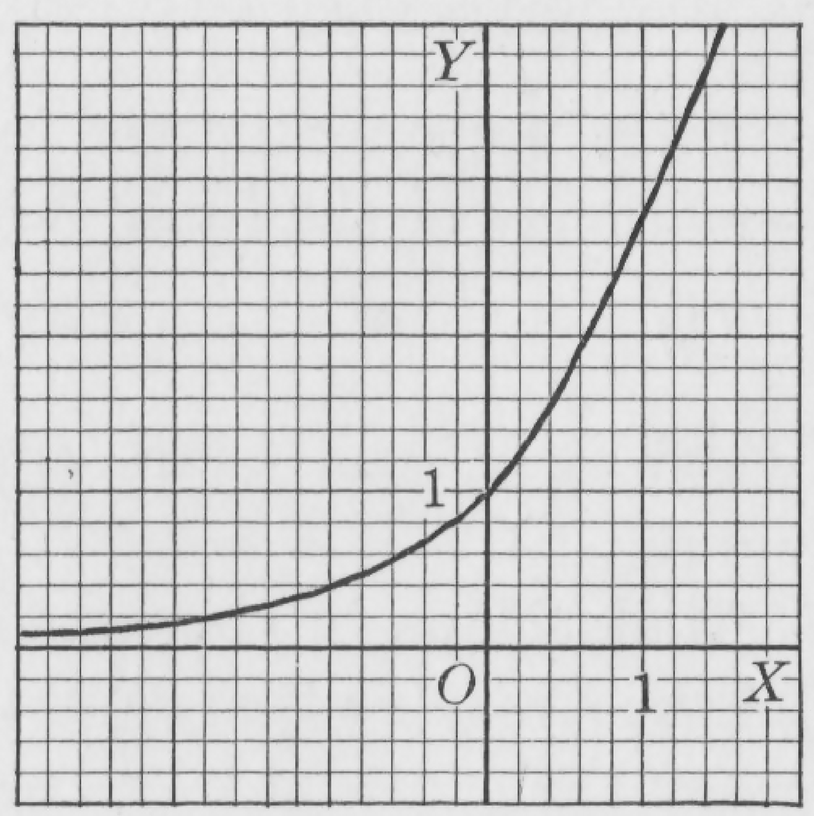


FIG. 86. *Exponential Curve.*

$$e^{-(x+1)} = e^{-1}e^{-x} = \frac{1}{2.72}e^{-x},$$

it follows that when x is given negative values which are larger and larger numerically, e^x is always positive and approaches zero. The curve approaches the negative x-axis as an asymptote.

The graph of

$$(2) \qquad y = \log_e x$$

can be obtained from the exponential curve in Figure 86. Since equation (2) is equivalent to

$$x = e^y,$$

we need only to interchange x and y in that Figure. The locus is shown in Figure 87.

If a quantity grows larger or smaller at a rate which is proportional always to its attained size, then its size is an exponential function of the time. For example, a sum of money A at interest at a rate r increases at compound interest according to the formula

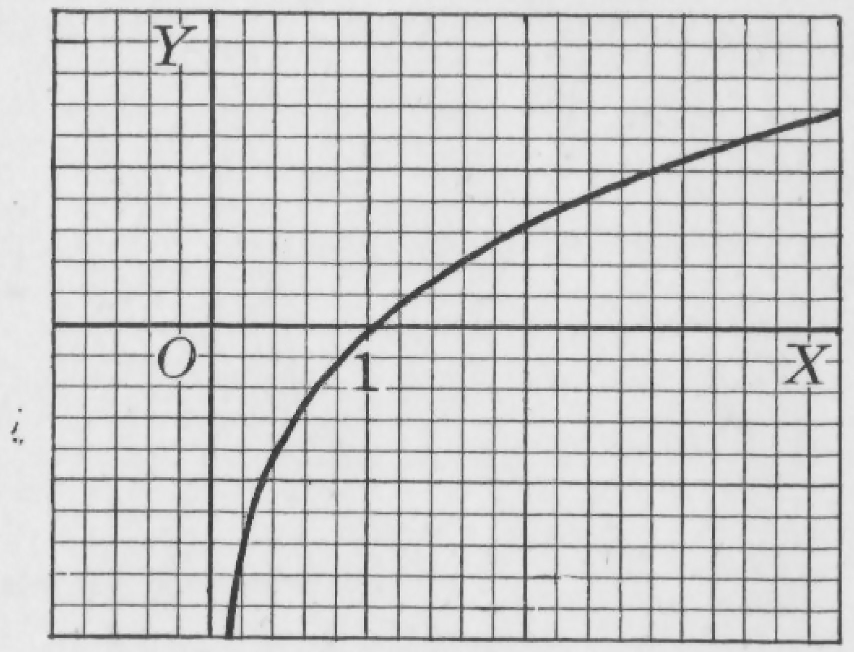


FIG. 87. *Logarithmic Curve*

$$A = P(1 + r)^n$$

where P is the initial principal and n is the number of years it has been at interest. If we determine k so that

$$1 + r = e^k, \quad k = \log_e(1 + r),$$

then we have

$$A = Pe^{kn}.$$

★ 80. Damped vibrations. If a tuning fork or string is set vibrating in air, the amplitude of its oscillations decreases, but the period remains practically constant. After t seconds the displacement of a point of the fork or string from its central position is given by an equation of the form

$$(1) \qquad y = ae^{-kt} \sin (\omega t + \alpha)$$

where a, k, ω, and α are constants. We have what is known as a *damped vibration.*

As an illustration of graphs of equations of this type, let us plot the graph of the special case where $a = 5$, $k = 0.1$, $\omega = 1$, $\alpha = 0$; this equation is

(2) $$y = 5e^{-(0.1)t} \sin t.$$

The curve is shown in Figure 88. We have drawn the boundary curves

(3) $$y = 5e^{-(0.1)t} \text{ and } y = -5e^{-(0.1)t}.$$

Since $\sin t$ varies from -1 to 1, the curve (2) oscillates between the curves (3), the amplitude approaching zero.

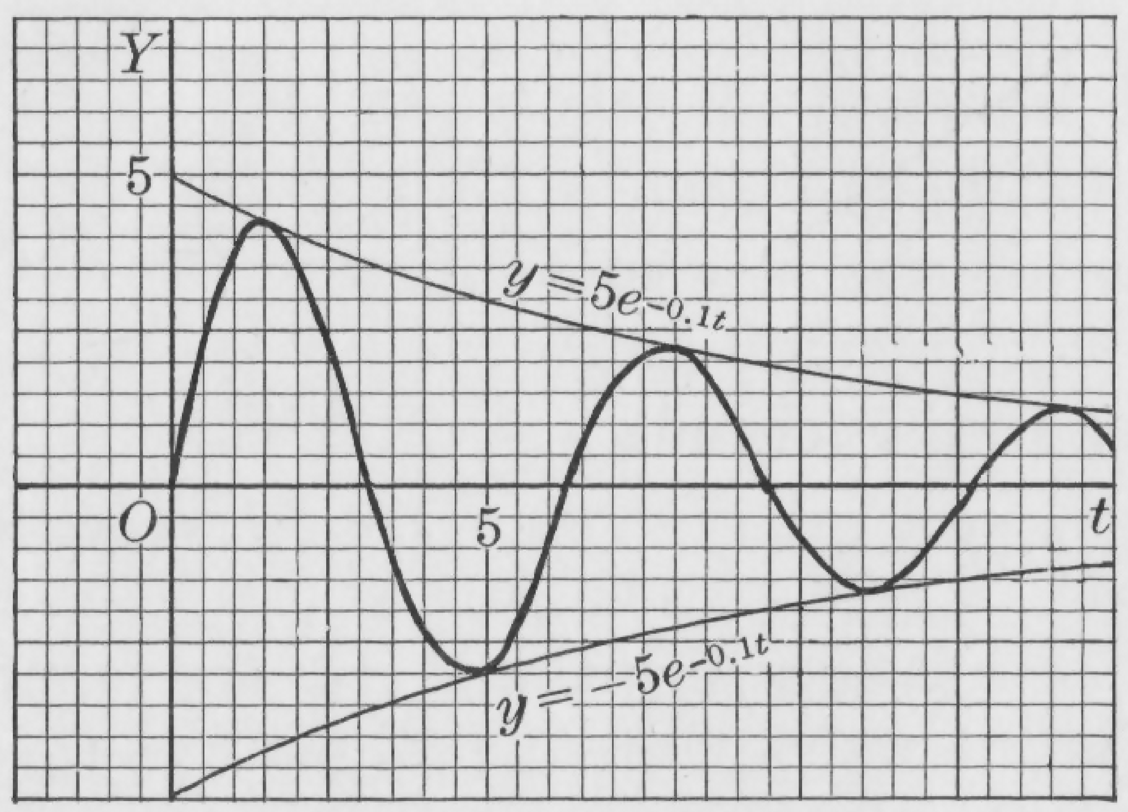


FIG. 88

EXERCISES

Plot the graph of each of the following equations.

1. $y = e^{(0.1)x}$.

2. $y = 10e^{-(0.1x)}$.

3. $y = e^{-(0.1)x^2}$.

4. $y = xe^{-x}$.

5. $y = x^2e^{-x}$.

6. $y = \log_{10} x$.

7. $y = x \log_e x$.

8. $y = \log_{10} (1 + x)$.

9. $y = \log_e (1 + x^2)$.

10. $y = e^{\log_e x}$.

11. $y = \sin (\sin^{-1} x)$.

12. $y = 4e^{-(0.2)x} \sin \pi x$.

13. $y = 4e^{-(0.1)x} \sin (\pi x + \pi)$.

14. $y = -5e^{(0.2)x} \cos \dfrac{\pi x}{2}$.

15. $y = \log \sin^2 \dfrac{\pi x}{3}$.

16. $y = x^2 \sin^2 x$.

17. $y = x - e^{(0.1)x}$.

18. $y = (1.06)^x$.

POLAR COÖRDINATES

81. Tracing a curve in polar coördinates.* Graphs in polar coördinates were obtained for simple equations on pages 26–30. We now proceed to a fuller discussion of the subject.

It should be recalled that a point may be described in polar coördinates in infinitely many ways. Thus $\left(r, \dfrac{\pi}{4}\right)$, $\left(r, \dfrac{\pi}{4} + 2n\pi\right)$, $\left(-r, \dfrac{\pi}{4} + \pi + 2n\pi\right)$, where n is a positive or negative integer, are polar coördinates of the same point.

The locus of an equation in polar coördinates contains every point such that at least one pair of coördinates of the point satisfies the equation, and it contains no other points. It may happen that only one of the infinitely many pairs of coördinates of a point satisfies the equation.

To trace a curve in polar coördinates we use much the same methods as were used with rectangular coördinates. It is desirable to test the curve for (1) intercepts, (2) symmetry, and (3) extent.

(1) *Intercepts.* The intercepts with the polar axis or this axis produced are found by substituting $\theta = 0, \pi, 2\pi, \cdots, -\pi, -2\pi, \cdots$ in the equation and solving for r. The curve passes through the origin if for some value of θ we have $r = 0$.

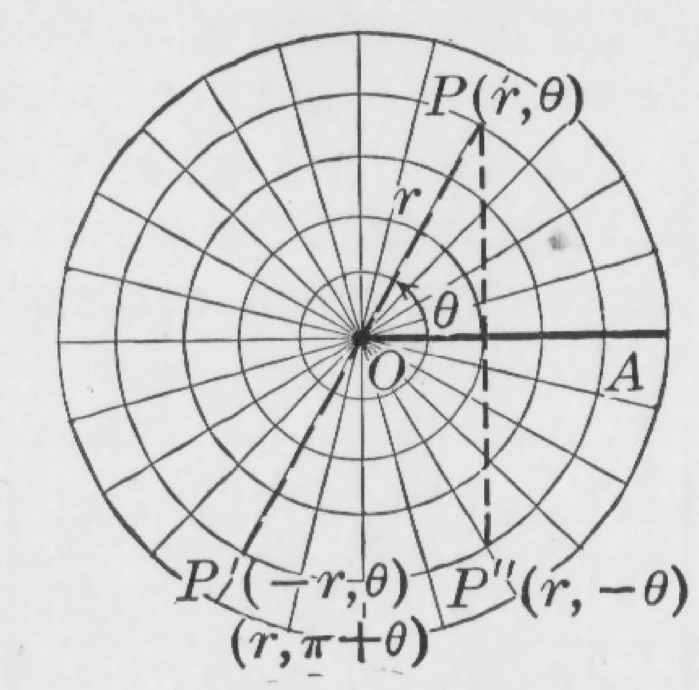


FIG. 89

(2) *Symmetry.* The curve is symmetrical with respect to the pole if whenever the equation is satisfied by (r, θ) it is also satisfied by $(-r, \theta)$, or by $(r, \pi + \theta)$, or by any other possible coördinates of the point $P'(-r, \theta)$. It is symmetrical with respect to the polar

* It is desirable to use paper especially ruled for polar coördinates.

axis if whenever the equation is satisfied by (r, θ) it is also satisfied by $(r, -\theta)$, or by $(-r, \pi - \theta)$, or by any other possible coördinates of $P''(r, -\theta)$.

(3) *Extent.* If the equation is solved for r in terms of θ it is often possible to determine values of θ, (a) which give a maximum or minimum value to r, (b) which make r become infinite, and (c) which make r imaginary. These things indicate the extent of the curve.

It is usually desirable to calculate values of one variable for a few values of the other.

It is sometimes simplest to change the equation to rectangular coördinates by the substitutions

$$x = r\cos\theta, \qquad y = r\sin\theta,$$
$$r^2 = x^2 + y^2, \quad \tan\theta = \frac{y}{x}.$$

Example 1. — Plot and discuss $r = a \sin 2\theta$, the four-leaved rose.

Solution. — (1) When $\theta = 0, \pi, 2\pi, \cdots, -\pi, -2\pi, \cdots$, we have $r = 0$. The curve intercepts the polar axis only at the pole. The pole is on the curve.

(2) The equation is altered by substituting $(-r, \theta)$ for (r, θ), but not by substituting $(r, \pi + \theta)$ for (r, θ), since $\sin(2\pi + 2\theta) = \sin 2\theta$; hence the curve is symmetrical with respect to the pole. The equation is altered by substituting $(r, -\theta)$, but not by substituting $(-r, \pi - \theta)$, since $\sin(2\pi - 2\theta) = -\sin 2\theta$; hence the curve is symmetrical with respect to the polar axis.

(3) The values of $\sin 2\theta$ vary from -1 to 1; hence r has values only between $-a$ and a. No value of θ makes r imaginary.

Consider the variation of r as θ increases continuously. It turns out to be convenient to divide the discussion into parts in each of which 2θ increases through $\pi/6$ radians. Results are shown in the following table.

2θ	0 to $\pi/6$	$\pi/6$ to $2\pi/6$	$2\pi/6$ to $3\pi/6$	$3\pi/6$ to $4\pi/6$	$4\pi/6$ to $5\pi/6$
θ	0 to $\pi/12$	$\pi/12$ to $2\pi/12$	$2\pi/12$ to $3\pi/12$	$3\pi/12$ to $4\pi/12$	$4\pi/12$ to $5\pi/12$
r	0 to $.5a$	$.5a$ to $.87a$	$.87a$ to a	a to $.87a$	$.87a$ to $.5a$

The student should extend the table up to $\theta = 2\pi$. Since $\sin 2(\theta + 2\pi) = \sin 2\theta$, the curve is retraced whenever θ passes through a complete

revolution. The curve is a **four-leaved rose** shown in Figure 90. When we increase θ continuously, the moving point describes the curve in the order indicated by the arrows and the numbers.

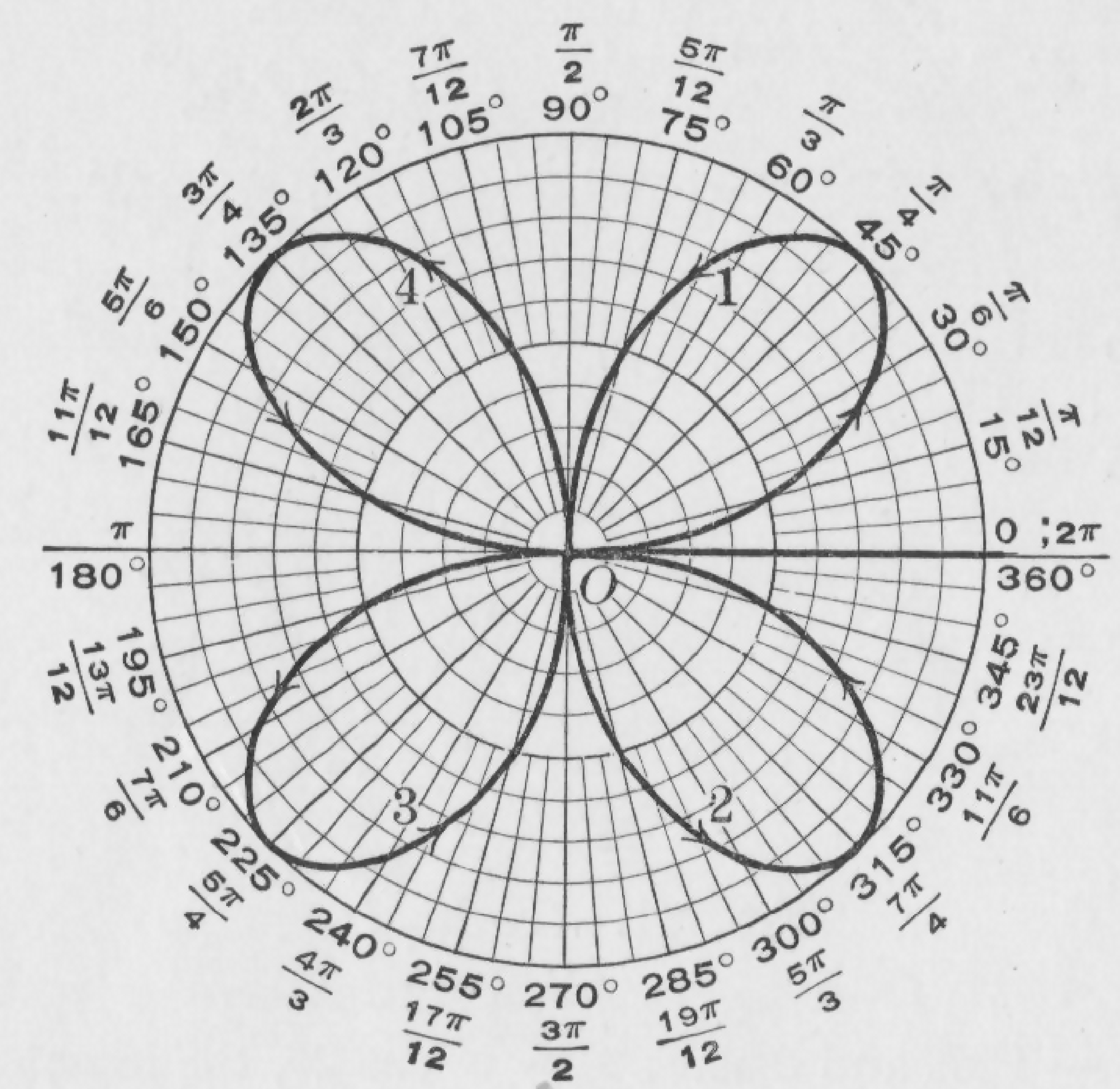


FIG. 90

Example 2. — Discuss and plot $r = a(1 - \cos\theta)$, the **cardioid.**

Solution. — (1) When $\theta = 0$, $r = 0$; when $\theta = \pi$, $r = 2a$. There are no other intercepts on the polar axis. When $r = 0$, $\cos\theta = 1$, and hence $\theta = 0 + 2n\pi$, where n is an integer.

(2) The equation is altered by substituting $(-r, \theta)$, or $(r, \pi + \theta)$, for (r, θ); the curve is, in fact, not symmetrical with respect to the pole. It is, however, symmetrical with respect to the polar axis, since the equation is unaltered if $(r, -\theta)$ is substituted for (r, θ).

(3) Since $\cos\theta$ varies from -1 to 1, r varies from $2a$ to 0. The value of r is real for every value of θ.

A set of points on the curve is given in the following table.

θ	0	$\pi/6$	$\pi/3$	$\pi/2$	$2\pi/3$	$5\pi/6$	π	$7\pi/6$
$\cos\theta$	1	.87	.50	0	$-.50$	$-.87$	-1	$-.87$
$1 - \cos\theta$	0	.13	.50	1	1.50	1.87	2	1.87
r	0	$.13a$	$.50a$	a	$1.50a$	$1.87a$	$2a$	$1.87a$

The graph is shown in Figure 91, page 178.

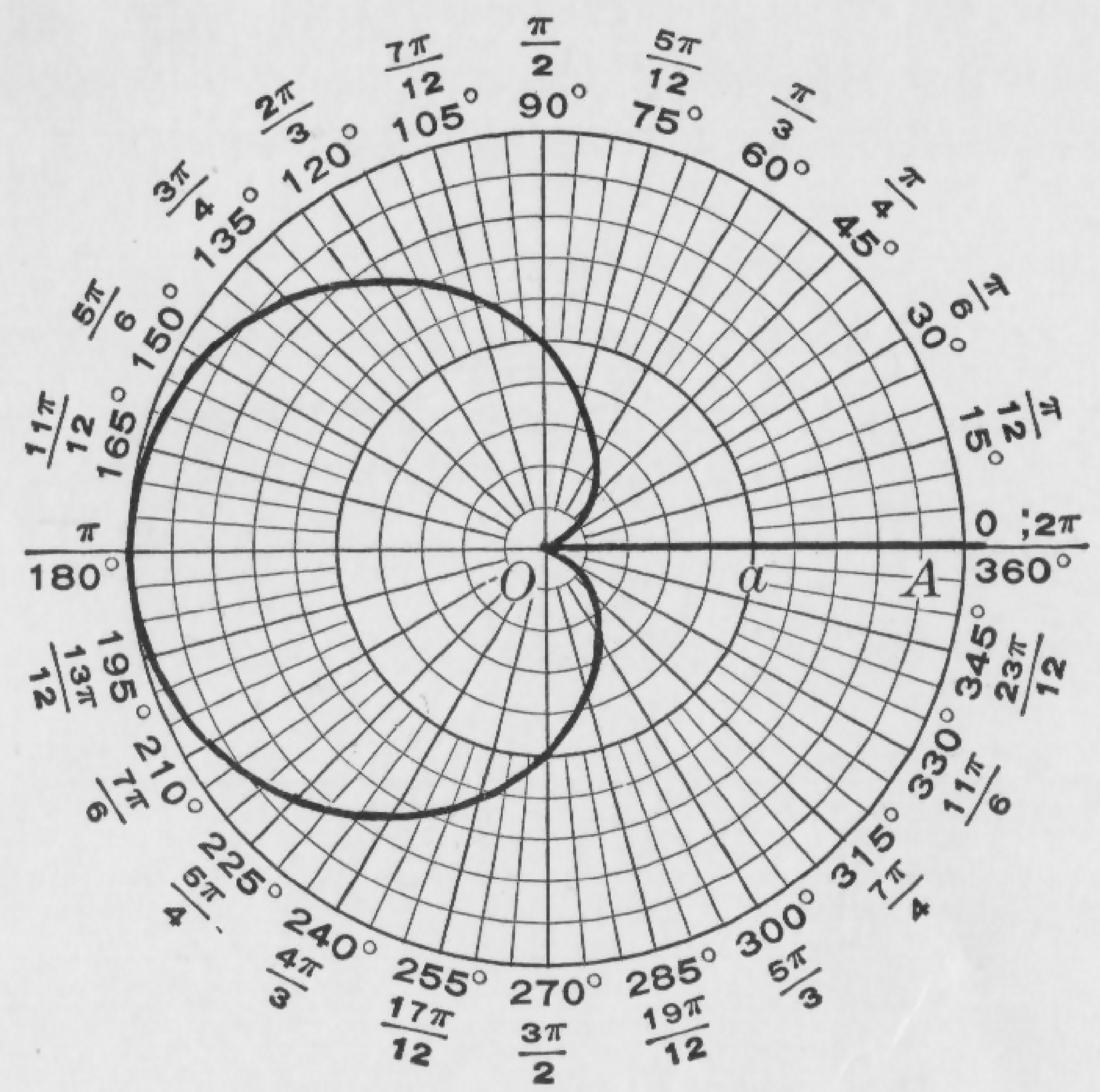


FIG. 91. *Cardioid*

Example 3. — Plot and discuss $r^2 = a^2 \cos 2\theta$, the **lemniscate.**

Solution. — (1) When $\theta = 0$, $r = \pm a$; when $\theta = \pi$, $r = \pm a$; there are no other intercepts on the polar axis. When $r = 0$, we have $2\theta = \frac{\pi}{2} + n\pi$, $\theta = \frac{\pi}{4} + n\frac{\pi}{2}$, where n is any integer; the curve passes through the pole for $\theta = \pi/4, 3\pi/4, 5\pi/4, 7\pi/4, \cdots$.

(2) The equation is unaltered by substituting $(-r, \theta)$, or $(r, -\theta)$, for (r, θ). Hence the curve is symmetrical with respect to the pole and to the polar axis.

(3) The values of r vary from $-a$ to a. The value of r is imaginary whenever $\cos 2\theta$ is negative, that is, when 2θ lies between $\pi/2$ and $3\pi/2$ or between $\frac{\pi}{2} + 2n\pi$ and $\frac{3\pi}{2} + 2n\pi$, where n is an integer; hence it is imaginary when θ lies between $\pi/4$ and $3\pi/4$, or $\frac{\pi}{4} + n\pi$ and $\frac{3\pi}{4} + n\pi$. There is no part of the curve corresponding to these angles.

A table of values and the graph follow.

2θ	0	$\pi/6$	$\pi/3$	$\pi/2$	$\pi/2$ to $3\pi/2$	$3\pi/2$	$5\pi/3$	$11\pi/6$	2π
θ	0	$\pi/12$	$\pi/6$	$\pi/4$	$\pi/4$ to $3\pi/4$	$3\pi/4$	$5\pi/6$	$11\pi/12$	π
r	$\pm a$	$\pm .9a$	$\pm .7a$	0	imag.	0	$\pm .7a$	$\pm .9a$	$\pm a$

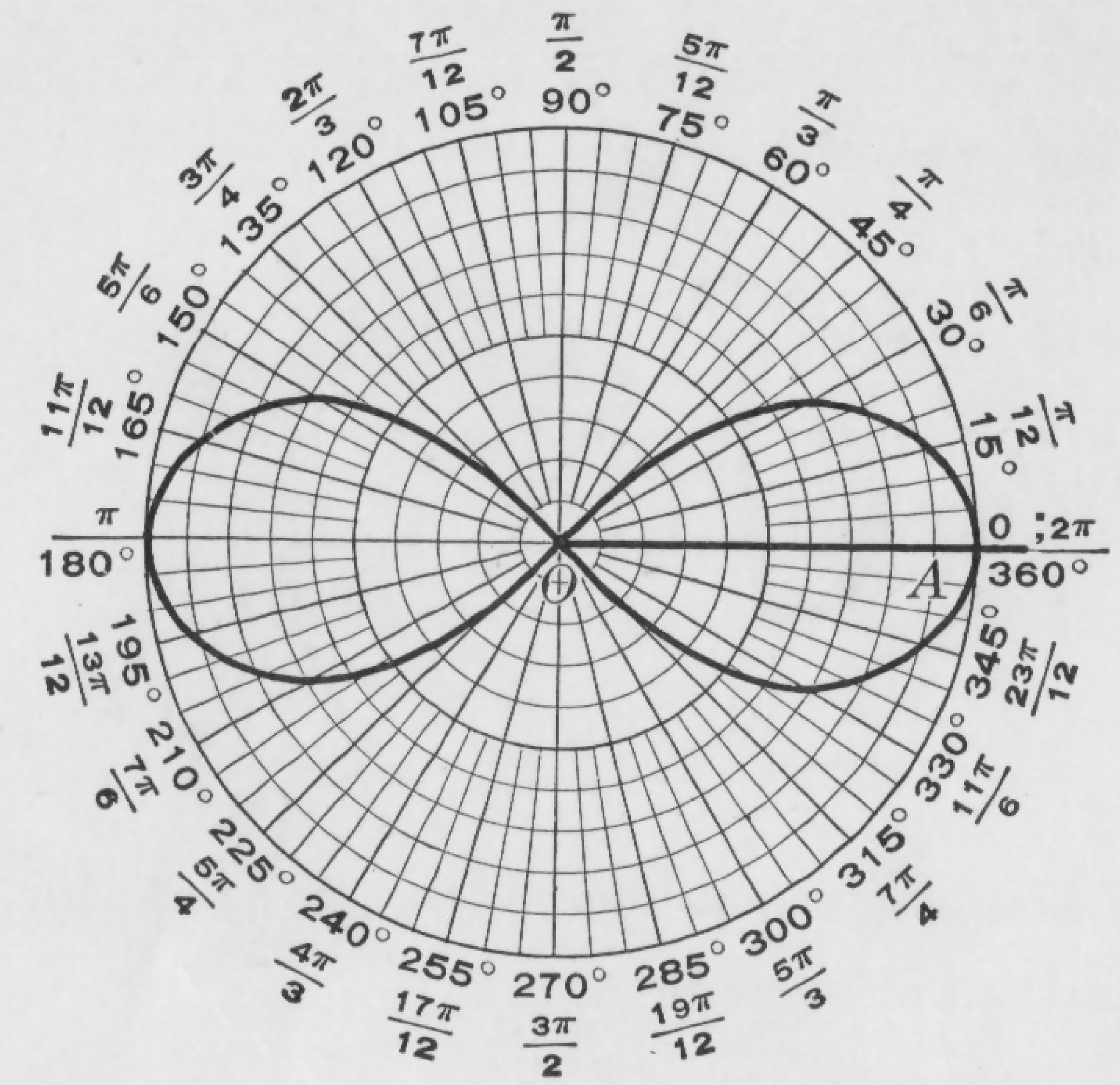


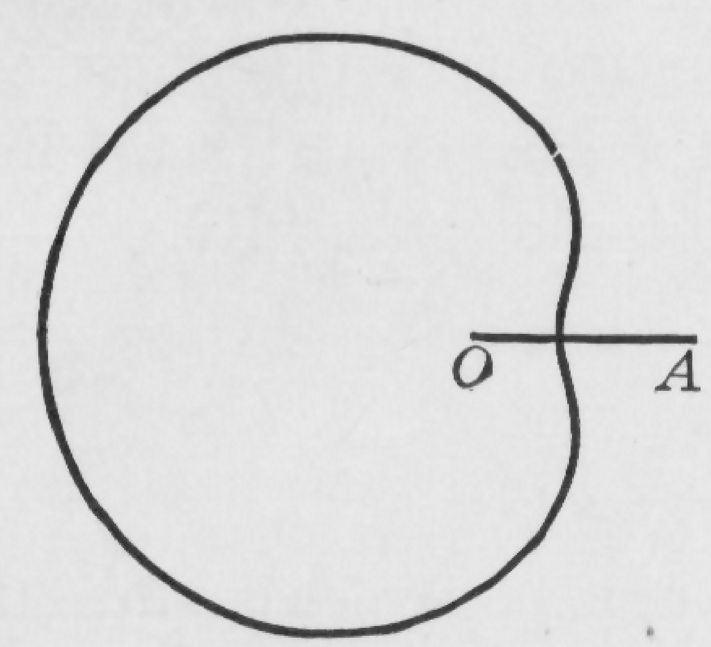
FIG. 92. *Lemniscate*

EXERCISES

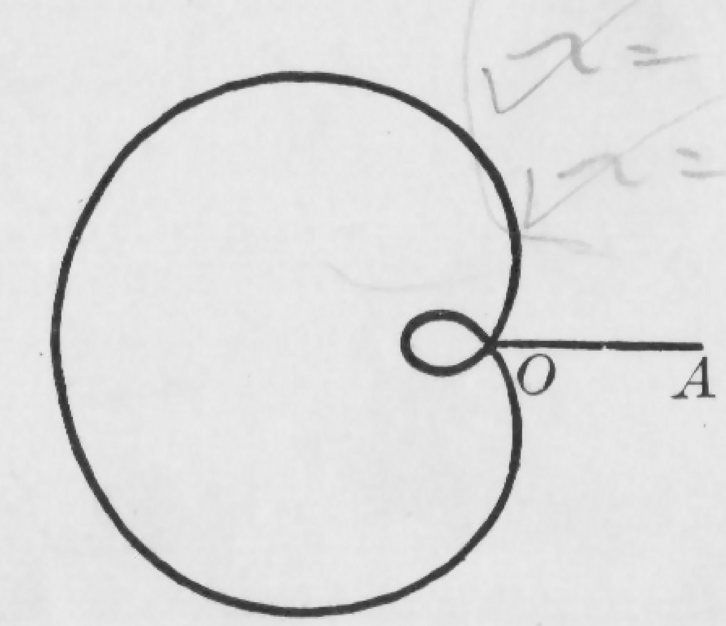
Discuss and plot each of the following equations.

1. $r = 10 \cos \theta$.
2. $r = 10 \sin \theta$.
3. $r = 10 \cos 2\theta$. 4 pet
4. $r = 10 \sec \theta$.
5. $r = 10 \csc \theta$.
6. $r = 10 \cos \theta + 5$. limacon
7. $r = \dfrac{10}{1 - \sin \theta}$.
8. $r = 10 \sin 3\theta$.
9. $r = 10 \cos 3\theta$. 3 pet
10. $r = 10 \sin 4\theta$. 8 petals
11. $r = 10 \cos 4\theta$.
12. $r = 10 \cos 5\theta$.
13. $r^2 = a^2 \cos 3\theta$.
14. $r = a \sin \frac{1}{2}\theta$.
15. The *limaçons*, $r = a - b \cos \theta$.

$r = 2 + 2 \sin \theta$
$r = 3 - 2 \cos \theta$
$r = 3 + 2 \cos \theta$



Limaçon, $a > b$



Limaçon, $a < b$

16. The *cissoid of Diocles*, $r = \dfrac{2a \sin^2 \theta}{\cos \theta}$.

17. The *spiral* $r = a\theta$.

18. The *spiral* $r = ae^{\theta}$.

19. The *spiral* $r\theta = a$.

20. The *spiral* $r^2 = a\theta$.

21. The ~~*lituus*~~ spiral $r^2\theta = a$.

22. The *conchoid of Nicomedes*, $r = a \sec \theta \pm b$.

23. The *ovals of Cassini* $(r^2 + a^2)^2 = 4a^2r^2 \cos^2 \theta + k^2$.

24. $r^2 + \theta^2 = \pi^2$.

Change the following equations into rectangular coördinates.

25. The equations of Exercises 1, 2, 3, 4.

26. The equations of Exercises 5, 6, 7.

27. The equations of Exercises 15, 16.

28. The equations of Exercises 22, 23.

Change each of the following equations into polar coördinates and plot the curve.

29. $(x^2 + y^2)^2 - 4x(x^2 + y^2) + 4x^2 = 16$.

30. $(x^2 + y^2)(x - 4)^2 = 4x^2$. A *conchoid.*

31. $(x^2 + y^2 + 4y)^2 = 4(x^2 + y^2)$. A *limaçon.*

32. $(x^2 + y^2)^2 = 4x(x^2 - 3y^2)$. A *three-leaved rose.*

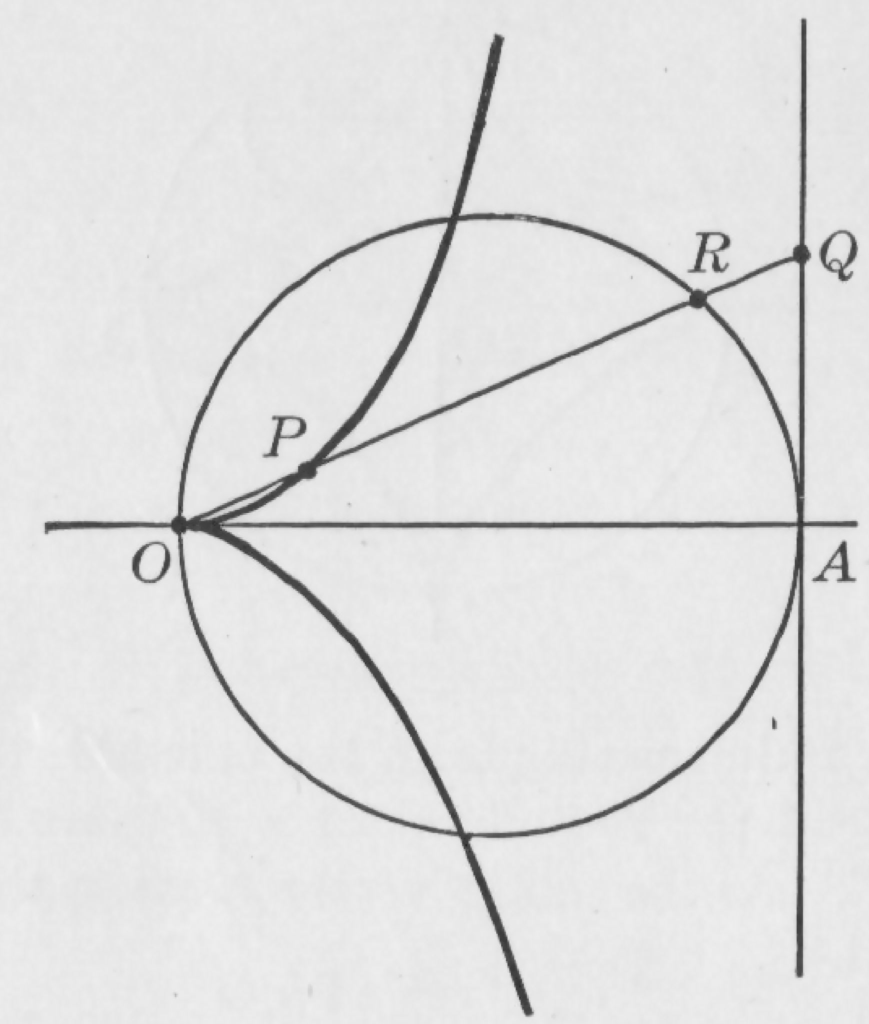
★ 82. Finding the polar equation of a curve. When a curve is described by geometrical conditions, how do we find a polar equation of the curve? The problem is quite similar to that of finding the equation in rectangular coördinates. It has been solved for the case of a straight line (page 69), a circle (page 105), and a conic (page 140).

EXERCISES

Find a polar equation of each of the following loci.

1. *Conic.* A point moves so that the ratio of its distance from the pole to its distance from the line $r \sin \theta = p$ is the constant e.

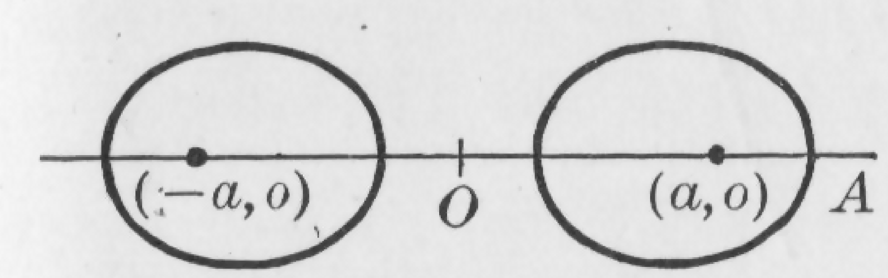
2. *Cissoid.* A secant OQ cuts the line $r \cos \theta = 2a$ at Q, and the circle $r = 2a \cos \theta$ at R. The point P, such that $OP = RQ$, describes the locus as the secant rotates about the point O.



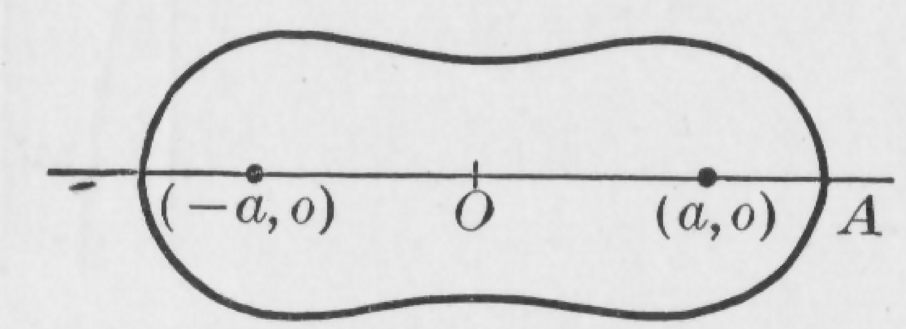
Cissoid

3. *Lemniscate.* A point moves so that the product of its distances from the points $(a, 0)$ and (a, π) is a^2.

4. *Oval of Cassini.* A point moves so that the product of its distances from the points $(a, 0)$ and $(-a, 0)$ is k^2.



Oval of Cassini, $k < a$

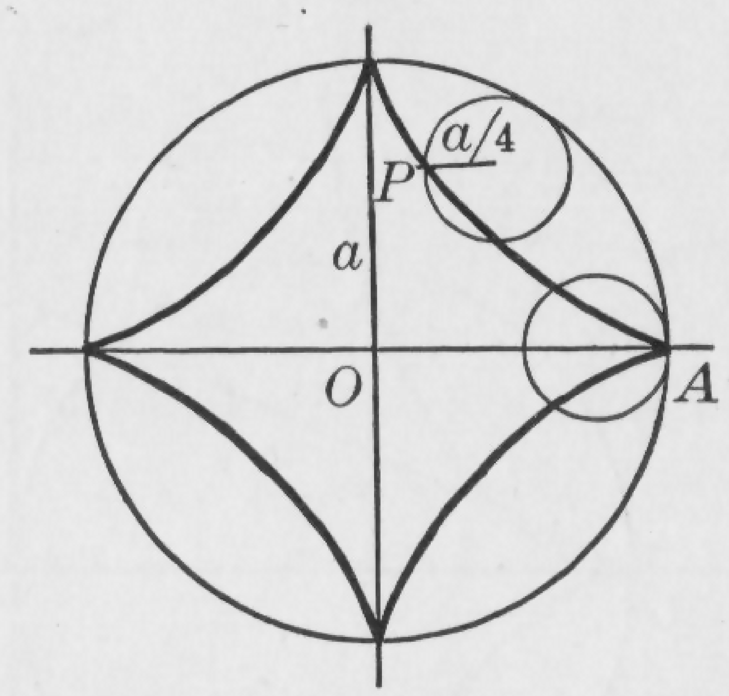


Oval of Cassini, $k > a$

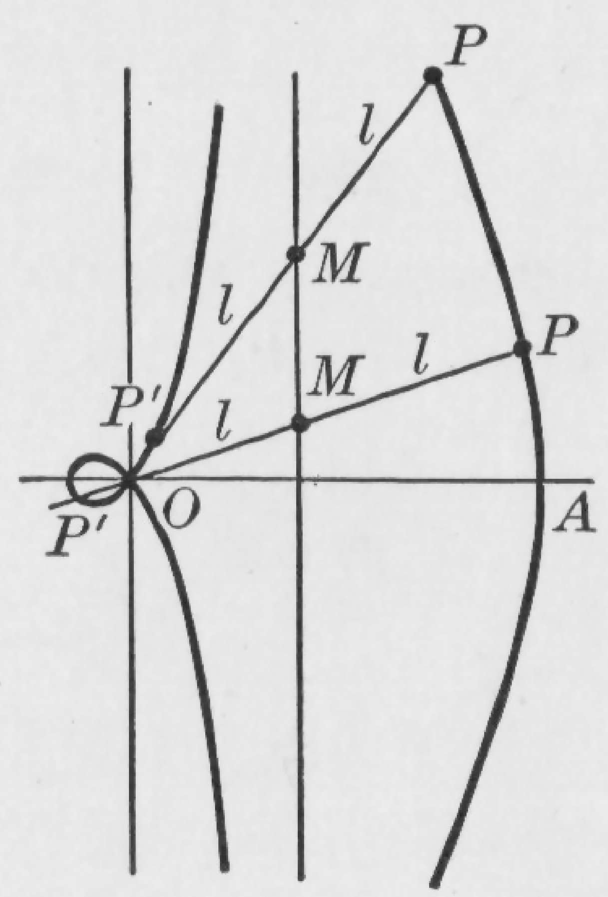
5. *Cardioid.* A secant OP cuts the circle $r = 2a \cos \theta$ at B, and $BP = 2a$. The point P describes the locus as OP rotates about the pole O.

6. *Limaçon.* A secant OP cuts the circle $r = 2b \cos \theta$ at B, and $BP = 2a$. The point P describes the locus as OP rotates about the pole O. If $b = a$ the limaçon is a cardioid.

7. *Four-cusped hypocycloid.* In the interior of the circle $r = a$ there is a circle of radius $a/4$ which rolls upon the larger circle. A point P of the small circle starts at the point $A(a, 0)$ and describes the curve.



8. *Conchoid.* A radius vector from the pole cuts the line $r \cos \theta = a$ at M; from M extend the radius vector a distance l in each direction to points P and P'. As the radius vector rotates, the points P and P' describe a conchoid.



★ 83. Equivalent equations in polar coördinates. Two equations are *analytically equivalent* if the solutions of each are solutions of the other. For example, the equations

$$2r = 6(\theta - 1), \quad r - 3\theta + 3 = 0,$$

are analytically equivalent.

Two equations in polar coördinates are *geometrically equivalent* if they have the same locus.

It follows at once that two equations which are analytically equivalent are also geometrically equivalent, but the converse is not always true. Consider, for example, the equations

$$r = \theta, \quad r = \theta + 2\pi.$$

They are not analytically equivalent, since, for example, $r = 1$, $\theta = 1$ is a solution of the former but not of the latter. But if $r = r_1$, $\theta = \theta_1$ is a solution of the former, then $r = r_1$, $\theta = \theta_1 - 2\pi$ is a solution of the latter, and the points (r_1, θ_1) and $(r_1, \theta_1 - 2\pi)$ coincide; it follows that the two equations have the same locus, and are therefore geometrically equivalent.

It is not difficult to see that similarly if we have an equation

$$f(r, \theta) = 0 \tag{1}$$

then the equations

$$f(r, \theta + 2n\pi) = 0, \tag{2}$$

$$f(-r, \theta + \pi + 2n\pi) = 0, \tag{3}$$

where n is an integer or zero, are geometrically equivalent to it. Of the infinitely many equations (2), (3), it often happens that all are analytically equivalent to one or two. For example, the equation

$$r = a(1 + \cos \theta) \tag{4}$$

and the equation obtained from it by replacing (r, θ) by $(-r, \theta + \pi)$, that is to say,

$$-r = a[1 + \cos(\theta + \pi)],$$

have the same locus. The latter reduces to

$$r = a(-1 + \cos \theta) \tag{5}$$

which is therefore geometrically equivalent to the former. The student should verify by plotting the graphs of the equations that the loci are the same; he should also verify that if formulas (2) and (3) are applied to equation (4), all other equations so obtained are analytically equivalent to (4) or (5).

★ 84. Intersections of curves whose equations are expressed in polar coördinates. If we solve simultaneously two equations in polar coördinates, and obtain a solution $r = r_1$, $\theta = \theta_1$, then (r_1, θ_1) satisfies both equations and is therefore a point of intersection of the corresponding curves.

If in place of one of the equations we take a geometrically equivalent equation and solve it simultaneously with the other equation, a solution gives also a point of intersection of the two curves.

To get all points of intersection of two curves having given polar equations we may need not only to solve the two given equations simultaneously, but also to solve other pairs of geometrically equivalent equations. We must also test directly to see if the pole is a point common to the two curves, the value of r being zero but that of θ indeterminate at this point.

Example. — Find the points of intersection of the curves

(1) $$r = \cos 2\theta, \quad r = 1 + \cos\theta.$$

Solution. — The steps in solving simultaneously are:

$$\cos 2\theta = 1 + \cos\theta,$$
$$2\cos^2\theta - 1 = 1 + \cos\theta,$$
$$2\cos^2\theta - \cos\theta - 2 = 0,$$
$$\cos\theta = \frac{1 \pm \sqrt{17}}{4} = 1.281 \text{ or } -0.781, \text{ approximately.}$$

Since $\cos\theta$ cannot exceed unity, we have only $\cos\theta = -0.781$; hence, approximately,

$$\theta = \pm 141° \pm n \cdot 360°.$$

It follows that

$$r = 1 - 0.781 = 0.219.$$

We thus obtain two distinct points

$$P_1(0.219, 141°), \quad P_2(0.219, -141°).$$

Equations which are geometrically equivalent to our first equation are found from equations (2) and (3) of § 83; we thus obtain

$$r = \cos 2(\theta + 2n\pi), \qquad -r = \cos 2(\theta + \pi + 2n\pi),$$

which reduce to

(2) $$r = \cos 2\theta, \quad r = -\cos 2\theta.$$

Equations which are geometrically equivalent to the second equation are

$$r = 1 + \cos(\theta + 2n\pi), \qquad -r = 1 + \cos(\theta + \pi + 2n\pi);$$

these reduce to

(3) $$r = 1 + \cos\theta, \quad r = -1 + \cos\theta.$$

The four pairs of equations to be solved simultaneously include, besides the pair (1) already solved,

(4) $$r = \cos 2\theta, \qquad r = -1 + \cos\theta;$$

(5) $$r = -\cos 2\theta, \quad r = 1 + \cos\theta;$$

(6) $$r = -\cos 2\theta, \quad r = -1 + \cos\theta.$$

Solutions of (4), (5) and (6) which give distinct points are found to be respectively

$Q_1(-1, 90°)$; $Q_2(-1, -90°)$; $Q_3(-0.5, 60°)$; $Q_4(-0.5, -60°)$;
$R_1(1, 90°)$; $R_2(1, -90°)$; $R_3(0.5, 120°)$; $R_4(0.5, -120°)$;
$S_1(-0.219, 39°)$; $S_2(-0.219, -39°)$.

Of the twelve points $P_1, \cdots, S_2$ only six are distinct, namely, $P_1, P_2, R_1, R_2, R_3, R_4$.

We see, finally, that the pole lies on both loci, for the coördinates $\left(0, \frac{\pi}{4}\right)$ satisfy the first of equations (1) and $(0, \pi)$ satisfy the second.

Thus there are seven points of intersection of the curves (1), as shown in Figure 93.

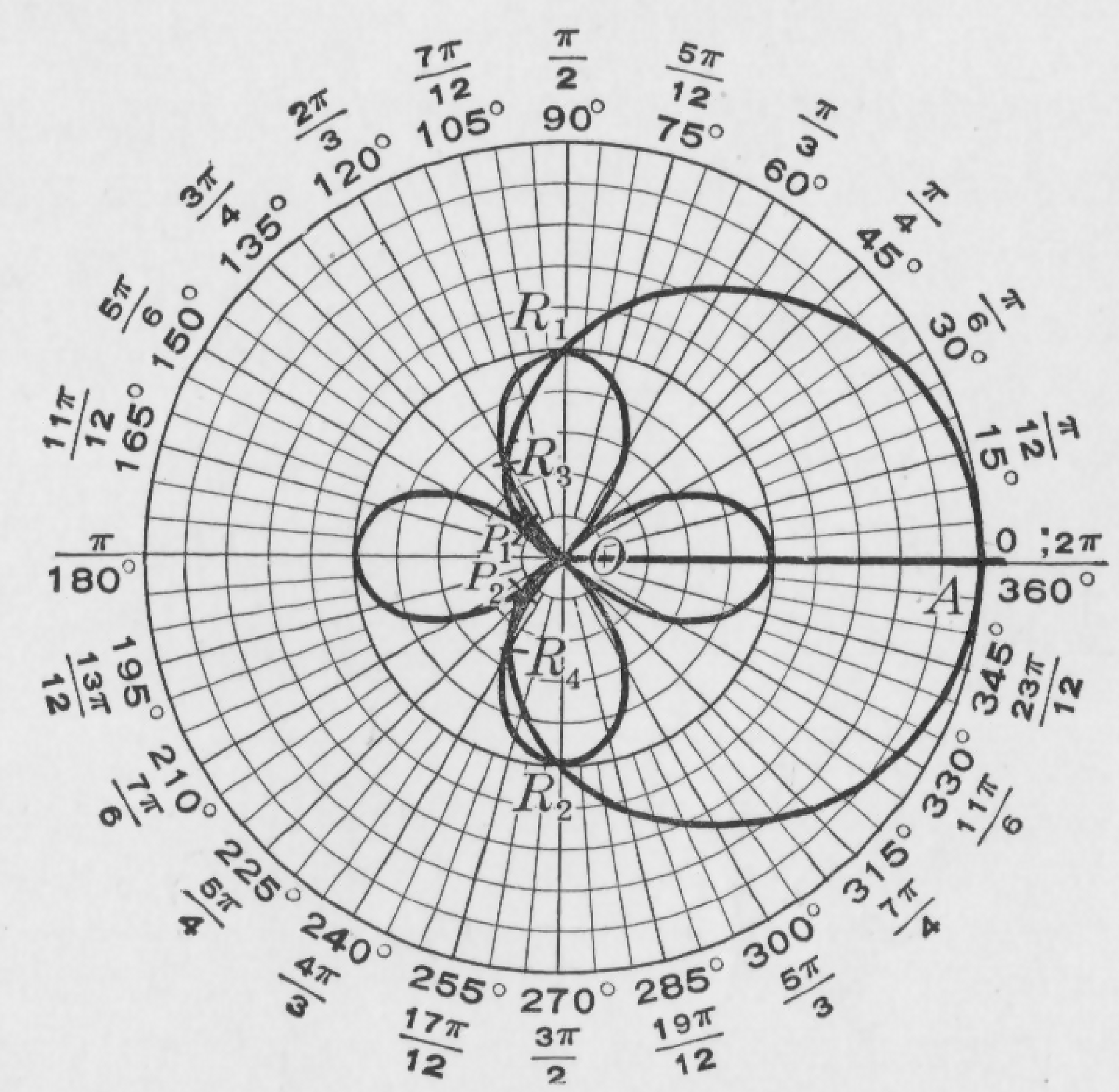


FIG. 93

EXERCISES

Find the points of intersection of the pair of curves in each of the following Exercises.

1. $r \cos \theta = 2,\ r = 4.$

2. $4r \cos \theta = 5,\ r = 5 \cos \theta.$

3. $r = 4 \sin \theta,\ r = 2\sqrt{3}.$

4. $r = 1 - \cos \theta,\ 2r = 1.$

5. $r = 2 \sin \theta,\ r = 2 \cos 2\theta.$

6. $r = 1 + \cos \theta,\ r(1 + \cos \theta) = 1.$

7. $r = \cos 2\theta,\ r = 1 + \sin \theta.$

8. $r = 4(1 - \sin \theta),\ r(1 + \sin \theta) = 3.$

9. $r^2 = 4 \cos 2\theta,\ r = \sqrt{8} \cos 2\theta.$

10. $r = \cos 3\theta,\ r = 4 \cos \theta.$

11. $r = \sin 3\theta,\ r = \sqrt{12} \sin 2\theta.$

12. $r = \theta,\ r = \theta^2.$

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CHAPTER IX

PARAMETRIC EQUATIONS

85. Parameters. Suppose we have two equations which express x and y in terms of a third variable t. For example,

$$(1) \qquad \begin{aligned} x &= 4 \cos t, \\ y &= 3 \sin t. \end{aligned}$$

If we give a value to t we determine one or more values of x and y. These corresponding values of x and y locate a point or points in a given rectangular coördinate system. The locus of all such points obtained by giving all possible values to t is called the locus of the **parametric equations,** t being the **parameter.** It is to be observed that the parameter is not one of the coördinates, but is another variable in terms of which the coördinates are expressed. It may or may not have an obvious geometric interpretation.

A curve given by parametric equations may be plotted by assigning a large number of values to the parameter and calculating the coördinates of corresponding points. Thus for equations (1) we form the following table.

t	0°	30°	60°	90°	120°	150°	180°	210°	240°	270°	300°	330°
$\cos t$	1	.87	.50	0	−.50	−.87	−1	−.87	−.50	0	.50	.87
$\sin t$	0	.50	.87	1	.87	.50	0	−.50	−.87	−1	−.87	−.50
x	4	3.5	2.0	0	−2.0	−3.5	−4	−3.5	−2.0	0	2.0	3.5
y	0	1.5	2.6	3	2.6	1.5	0	−1.5	−2.6	−3	−2.6	−1.5

Since $\sin t$ and $\cos t$ have a period of 360°, the values of x and y repeat after t has gone from 0° to 360°.

If we plot the points we find that they appear to lie on an ellipse. That they actually are on an ellipse may be shown by finding the equation of the curve obtained by eliminating the parameter. The equations may be written

$$\frac{x}{4} = \cos t, \quad \frac{y}{3} = \sin t.$$

We eliminate t by squaring and adding these equations; we get

$$\frac{x^2}{16} + \frac{y^2}{9} = 1,$$

the equation of an ellipse.

By referring to § 59, page 138, we observe that in these equations (1) the parameter t is the eccentric angle for the ellipse.

Develop

86. Path of a projectile. Suppose a projectile is fired from a gun and moves subject to no force except the constant downward attraction of the earth. Let v be the muzzle velocity and α the angle of elevation of the gun. Take the position of the gun as origin of rectangular coördinates, the y-axis vertical, and the x-axis so that the path of the projectile lies in the first quadrant of the XOY plane (Fig. 94).

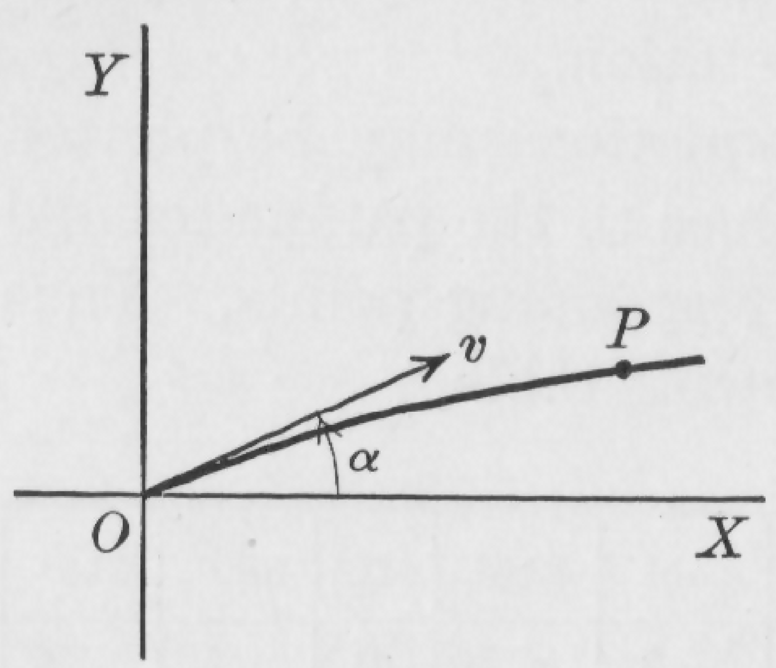


FIG. 94

The x-component of the initial velocity is $v \cos \alpha$. Since there is no component of force in the x-direction, the projectile moves with uniform velocity in that direction. Hence the abscissa of the projectile t seconds after the gun is fired is

$$(1) \qquad x = (v \cos \alpha)t.$$

The y-component of the initial velocity is $v \sin \alpha$. If there were no component of force in the y-direction, the height of the projectile after t seconds would be $(v \sin \alpha)t$. The attraction of the earth, however, decreases this by $\frac{1}{2}gt^2$, where g is a constant. Hence the ordinate of the projectile is

$$(2) \qquad y = (v \sin \alpha)t - \tfrac{1}{2}gt^2.$$

The equations (1) and (2) are parametric equations of the path of the projectile.* If we eliminate t we get

$$(3) \qquad y = x \tan \alpha - \frac{gx^2}{2v^2 \cos^2 \alpha},$$

which shows that the curve is a parabola whose axis is vertical.

EXERCISE

Plot each of the curves having the following parametric equations, and find an equation in x *and* y *for each by eliminating the parameter.*

1. $x = 2t, \quad y = 3t + 4.$
2. $x = t + 2, \quad y = t^2 - 4.$
3. $x = 6 \cos t, \quad y = 2 + 6 \sin t.$

1-9

4. $x = 3 + 5 \cos t, \quad y = 4 + 3 \sin t.$
5. $x = t^2, \quad y = t^3.$
6. $x = \sin^2 t, \quad y = \sin^3 t.$
7. $x = 5 \sec \theta, \quad y = 4 \tan \theta.$
8. $x = 10 \cos^3 \theta, y = 10 \sin^3 \theta.$ (*Hypocycloid of four cusps.*)
9. $x = \dfrac{3at}{1 + t^3}, y = \dfrac{3at^2}{1 + t^3},$ a being constant. (*Folium of Descartes.*)

10. Show that the highest point in the path of a projectile is the point $\left(\dfrac{v^2 \sin 2\alpha}{2g}, \quad \dfrac{v^2 \sin^2 \alpha}{2g}\right)$.

11. Show that the horizontal range of a gun, *i.e.*, the distance from gun to target in a horizontal plane, is $v^2 \sin 2\alpha/g$. For what angle of elevation of the gun is the range the greatest?

---

* It is easy to see that in problems in mechanics the time t will frequently occur as a parameter.

12. A rifle has a muzzle velocity $v = 2000$ feet per second. It is fired at an angle of elevation of 30°. Find the position of the projectile at the end of the fifth second after firing, and every five seconds thereafter until it drops to the level of the rifle. Plot the path. Take $g = 32$.

13. Suppose that in a given pair of parametric equations, for example

(a) $$x = a \cos t, \quad y = b \sin t,$$

we substitute for the parameter an expression involving some other parameter s, for example

$$t = s^2.$$

We thus have x and y expressed differently in terms of a parameter, for example

(b) $$x = a \cos s^2, \quad y = b \sin s^2.$$

Is the locus of these new equations the same as that of the original equations?

14. During naval target practice an observer noted that when a gun was fired nearly horizontally at a target a period of 5 seconds elapsed between the flash of the gun and the splash of the projectile as it hit the target; when the gun was fired nearly vertically at the same target there was 70 seconds between flash and splash. Assuming that the formulas of § 86 apply in this case, find the range, the muzzle velocity of the gun, and the greatest height of the projectile for each path.

87. The witch. Let us find equations of the following locus, known as the **witch of Agnesi.** Through the origin O of a system of rectangular coördinates draw a line of inclination α, cutting the circle $x^2 + (y - a)^2 = a^2$ at A, and the line $y = 2a$ at B. Through A and B draw lines parallel to the x- and y-axes respectively; let P be their point of intersection. As α varies from 0 to π, the point P describes the locus (Fig. 95).

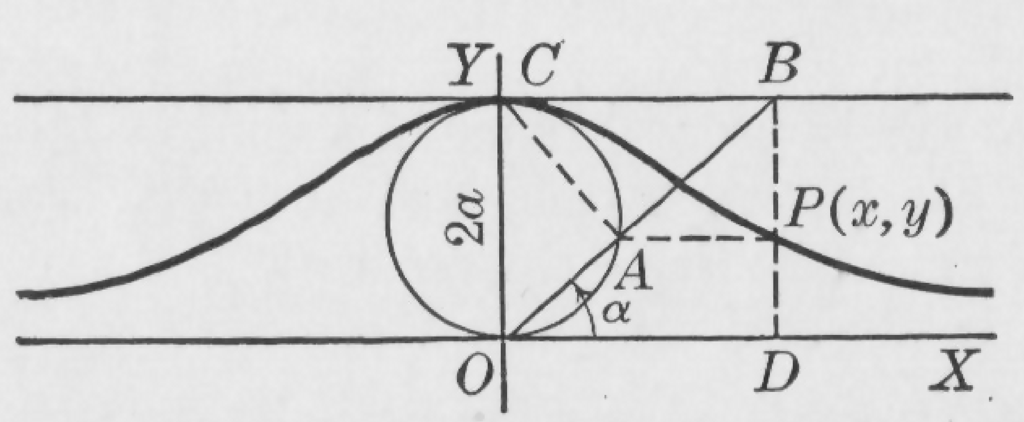


Fig. 95

Let (x, y) be the coördinates of P. In the figure, $OC = DB = 2a$, and we have at once, since $OD = x$ is the base of the right triangle ODB,

(1) $$x = 2a \cot \alpha.$$

The angle OAC is a right angle, and therefore we have $OA = 2a \sin \alpha$. It readily follows, since $y = OA \sin \alpha$, that

(2) $$y = 2a \sin^2 \alpha.$$

Thus (1) and (2) are parametric equations of the witch of Agnesi.

To eliminate the parameter α, we note that

$$x^2 + 4a^2 = 4a^2(\cot^2 \alpha + 1) = 4a^2 \csc^2 \alpha;$$

and, since $\sin \alpha \csc \alpha = 1$, we have

(3) $$y(x^2 + 4a^2) = 8a^3.$$

It is seen that the curve is symmetrical with respect to the y-axis, and that the x-axis is a horizontal asymptote.

88. The strophoid. Let AO be the perpendicular from a point A to a line BC. Through A draw a line of inclination α to AO, cutting BC at E. On this line locate P and P' such that

$$OE = EP = EP'.$$

The locus of P and P' as α varies from $-\pi/2$ to $\pi/2$ is called a **strophoid.**

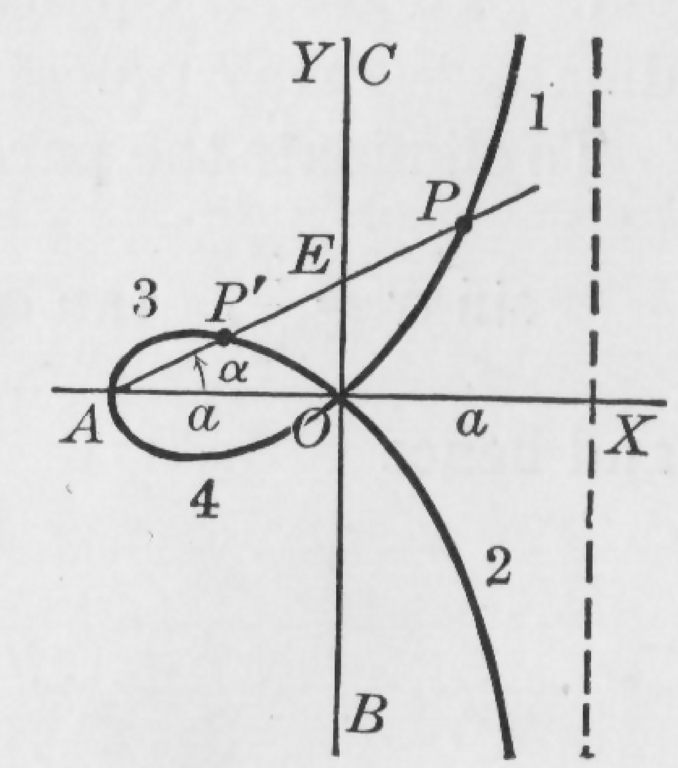


Fig. 96

To find equations for the curve, take O as origin of a set of rectangular coördinates, and let the coordinates of A be $(-a, 0)$. We see that BC is the y-axis, and that

$$OE = a \tan \alpha.$$

Let the coördinates of P be (x, y). We have at once

$$x = EP \cos \alpha = a \tan \alpha \cos \alpha = a \sin \alpha,$$
$$y = OE + EP \sin \alpha = a \tan \alpha(1 + \sin \alpha).$$

Thus parametric equations of the locus of P are

$$(1) \qquad x = a \sin \alpha, \quad y = a \tan \alpha (1 + \sin \alpha).$$

Let the coördinates of P' be (x', y'). We have

$$x' = -EP \cos \alpha = -a \sin \alpha$$
$$y' = OE - EP \sin \alpha = a \tan \alpha (1 - \sin \alpha).$$

Hence parametric equations of the locus of P' are

$$(2) \qquad x' = -a \sin \alpha, \quad y' = a \tan \alpha (1 - \sin \alpha).$$

We observe that if we substitute $\alpha = \pi + \alpha'$ in these equations they reduce to

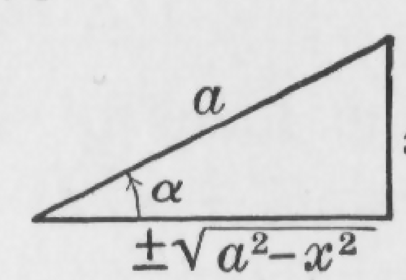
$$x' = a \sin \alpha', \quad y' = a \tan \alpha' (1 + \sin \alpha'),$$

which are of form (1). It follows that if we allow α to vary from 0 to 2π in equations (1), the total curve in Figure 96 will be described once and only once. The reader may verify the statement that portions of the curve are described in the order 1, 2, 3, 4 as indicated.

Thus equations (1) are parametric equations of the strophoid. To get an equation of the curve in rectangular coördinates we may proceed as follows.

To eliminate the parameter from (1) we have

$$\sin \alpha = \frac{x}{a}, \quad \tan \alpha = \frac{x}{\pm\sqrt{a^2 - x^2}},$$



and hence

$$y = \frac{ax\left(1 + \dfrac{x}{a}\right)}{\pm\sqrt{a^2 - x^2}}.$$

Squaring and simplifying, we get

$$(3) \qquad y^2 = x^2 \frac{a + x}{a - x}.$$

The curve (3) is symmetrical with respect to the x-axis, does not extend to the right of the line $x = a$, nor to the left of the line $x = -a$, has the line $x = a$ as a vertical asymptote, and is cut by a line parallel to the x-axis in at most three points.

89. The cycloid. A circle moving in a plane rolls along a straight line. A point on the circumference describes a curve called the **cycloid.**

To find equations for the cycloid, choose a rectangular coördinate system with the given line as x-axis, and such that the moving point passes through the origin and the rolling circle lies above the x-axis. Let a be the radius of the circle, and $P(x, y)$ the point which describes the curve. Let θ be the angle through which the circle has rolled after P has passed through the origin. Then, in the notation of the figure,

$$x = OA - DA.$$
$$y = AC - BC.$$

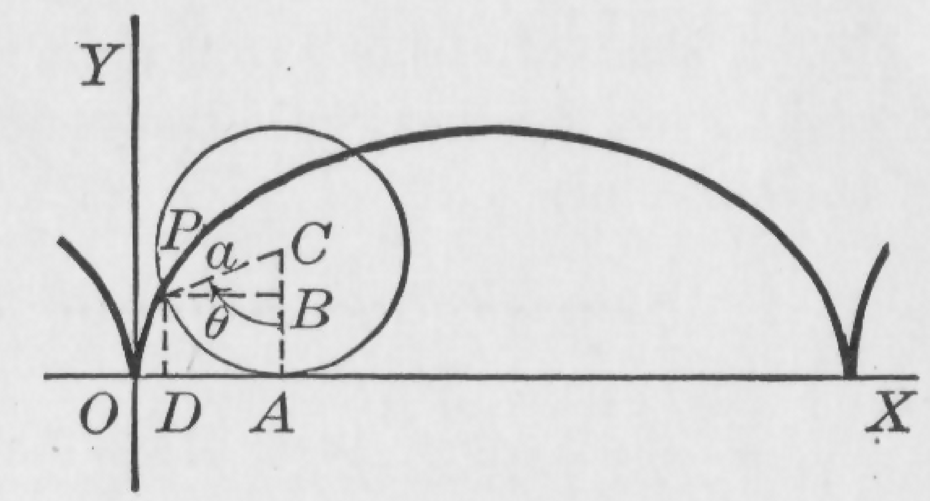


FIG. 97

Since the circle rolls,

$$OA = \text{arc } AP = a\theta.$$

We see that

$$DA = PB = a \sin \theta, \quad AC = a, \quad BC = a \cos \theta,$$

and hence that

$$(1) \qquad \begin{aligned} x &= a(\theta - \sin \theta), \\ y &= a(1 - \cos \theta). \end{aligned}$$

These are parametric equations of the cycloid. The elimination of θ gives the equation

$$(2) \qquad x = a \cos^{-1}\left(1 - \frac{y}{a}\right) \pm \sqrt{2ay - y^2}.$$

We note that this equation is *not algebraic* but is *transcendental,* involving a trigonometric function.

EXERCISES

1. In the equation of the folium of Descartes,

$$x^3 + y^3 = 3axy,$$

substitute $y = tx$, and find parametric equations, with t as parameter.

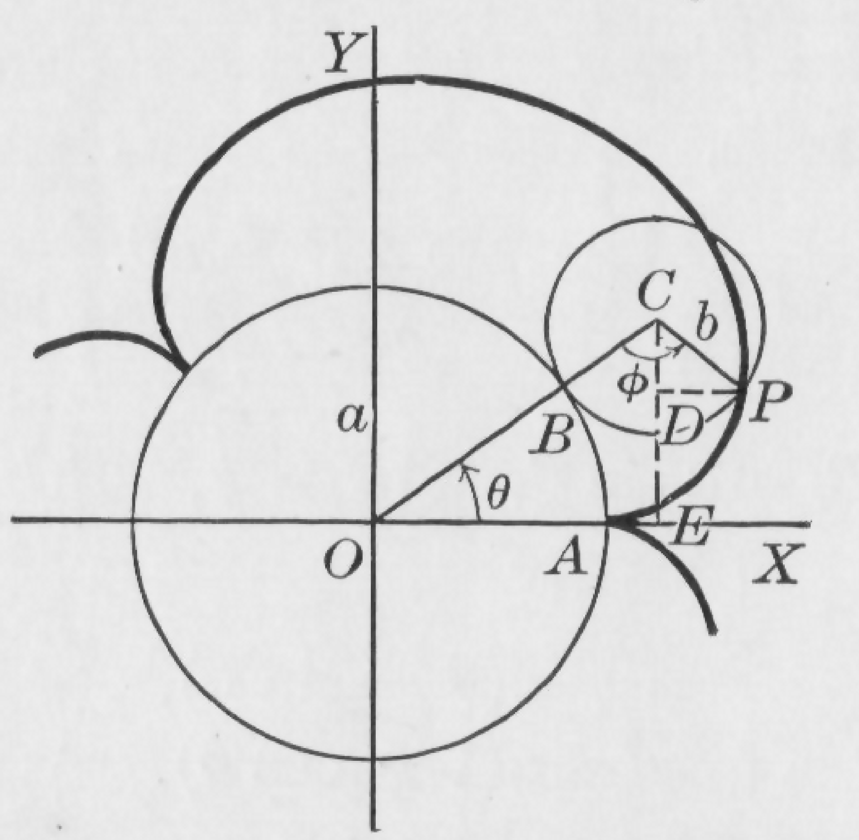
2. Find parametric equations of the cycloid if the origin is taken at the highest point of an arch, the cusps (pointed portions, as at O in Fig. 97) pointing downward.

3. A circle of radius a moving in a plane rolls along a straight line. Find equations of the locus of a point P on a radius of the circle at a distance l from the center. If $l > a$, the curve is called a **prolate cycloid;** if $l < a$, it is a **curtate cycloid.** These curves are also called **trochoids.**

4. A circle of radius b rolls upon the exterior of a circle of radius a. A point P on the first circle traces an **epicycloid.** Show that equations of the curve are

$$x = (a + b) \cos \theta - b \cos \frac{a + b}{b} \theta,$$

$$y = (a + b) \sin \theta - b \sin \frac{a + b}{b} \theta.$$



5. Find equations of the epicycloid (Ex. 4) when $a = b$. Draw the curve. Transfer the origin to the point A of the figure, and find an equation of the curve in polar coördinates. Show that the curve is a cardioid.

6. Find equations of the epicycloid (Ex. 4) and draw the figure

(*a*) when $a = 2b$; (*c*) when $a = 4b$;
(*b*) when $a = 3b$; (*d*) when $a = 5b$.

7. A circle of radius b rolls upon the interior of a fixed circle of radius a. A point P upon the first circle describes a **hypocycloid.** Show that equations of the curve are

$$x = (a - b) \cos \theta + b \cos \frac{a - b}{b} \theta,$$

$$y = (a - b) \sin \theta - b \sin \frac{a - b}{b} \theta.$$

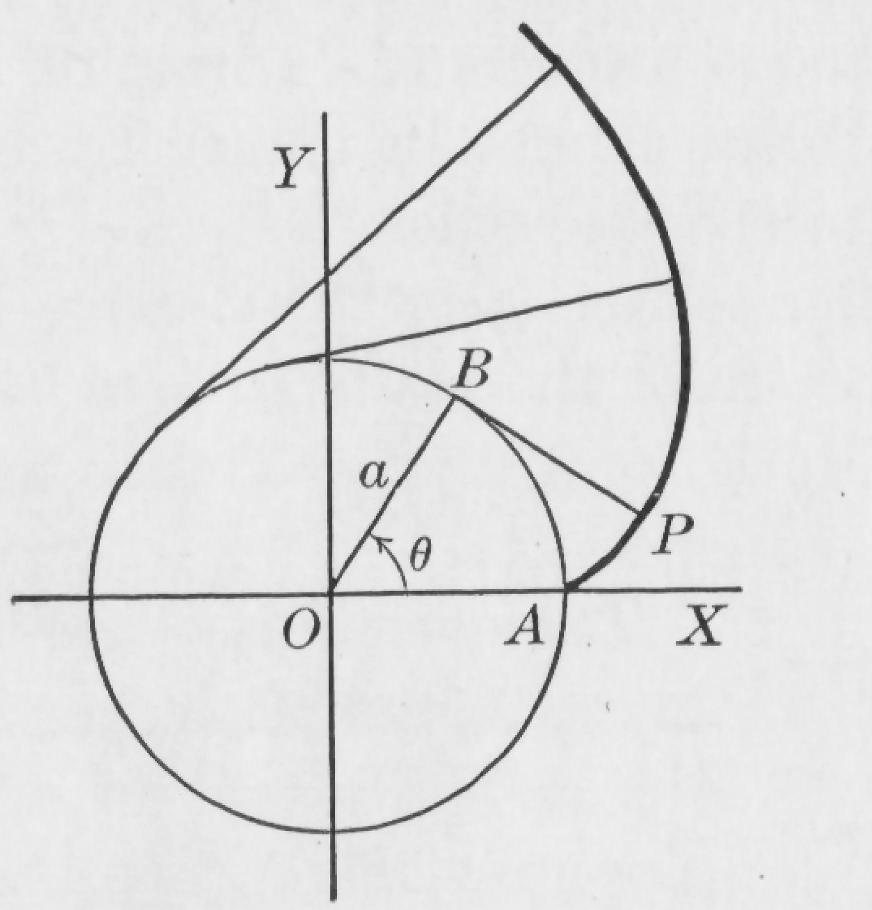
8. Find equations of the hypocycloid (Ex. 7), and plot the curve when

(*a*) $a = 2b$; (*b*) $a = 3b$.

9. Find equations of the **hypocycloid of four cusps** ($a = 4b$, Ex. 7). Plot the curve.

10. In unwinding a circular spool of thread which is held stationary the thread is kept taut and always in a fixed plane. The curve described by the point at the end of the thread is the **involute of a circle.** Show that equations of the curve are

$$x = a(\cos \theta + \theta \sin \theta);$$
$$y = a(\sin \theta - \theta \cos \theta).$$



11. A wheel of radius b rolls on the exterior of a fixed circle of radius a. A point P on a spoke of the wheel at a distance l from its center describes a locus. Find equations of the curve. Show that if $b = a$ and $l = 2b$ the polar equation is $r = 4b \sin (\phi/3)$, where (r, ϕ) are polar coördinates of P, the pole being at the center of the fixed circle, and the direction of the polar axis being suitably chosen.

Hint. In the special case where $b = a$ and $l = 2b$, obtain a polar equation, then change this to the required form by a transformation of polar coördinates which amounts to rotating the polar axis through a certain angle.

90. Notes on certain higher plane curves. A curve which lies in a plane but is not a straight line or a conic is sometimes referred to as a **higher plane curve.** In the preceding pages a number of such curves have been mentioned. We shall now state without proof some interesting facts concerning these curves.

The **cissoid** (page 181) was studied by Diocles, a Greek of the second century B.C. By means of this curve and a ruler and compass it is possible to "duplicate a cube," that is, to find the edge of a cube whose volume is twice that of a given cube. The duplication of a cube was one of the famous problems of antiquity. It cannot be solved by use of ruler and compass alone.

The **conchoid** (page 182) was studied by Nicomedes, another ancient Greek. Like the cissoid it can be used with ruler and compass to duplicate a cube, and also to solve the equally famous problem of "trisecting any given angle." The latter problem cannot be solved by ruler and compass alone except for special angles.*

The **Cassinian ovals** (page 181) were suggested as the shape of the orbits of the planets about the sun, instead of their actual elliptic form, by the Italian astronomer Cassini (1625–1712).

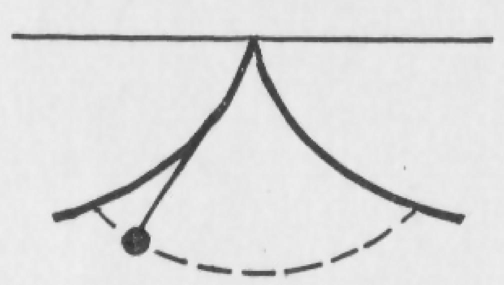


FIG. 98

The cycloid is obviously of importance in connection with the motion of points on the circumference of a wheel which rolls along a straight line. It is otherwise of interest also. Thus it is shown in mechanics that if a pendulum bob supported by a thread swings freely, the period of the swing depends upon the amplitude; but if it is supported at the cusp of a

* The third very famous problem of antiquity was that of "squaring a circle," that is, of finding by ruler and compass a square whose area equals that of a given circle. This cannot be solved by the use of those instruments alone, nor by their use in conjunction with the graph in rectangular coördinates of any algebraic equation whose coefficients are rational numbers.

certain cycloidal frame in such a way that it winds up slightly at each end of its swing (Fig. 98), then its period is independent of its amplitude, and the path of the bob is an arc of another cycloid.

The cycloid is also the curve of quickest descent; that is, if a bead slides without friction down a wire from a point A to a point B, the time required depends upon the shape of the wire, being least when it is an arc of a cycloid whose cusps point upward.

CHAPTER X

TANGENTS AND NORMALS

91. The tangent to a curve at a point on the curve. Let P_1 and P_2 be two points on a curve C. Draw the secant line P_1P_2. Now let P_2 move along the curve toward P_1. The secant line P_1P_2 turns about the point P_1 and as P_2 approaches coincidence with P_1, the line P_1P_2 approaches coincidence with a limiting line P_1T. This limiting line is called the **tangent line** to the curve C at the point P_1. The point P_1 is the **point of contact** of the tangent line.

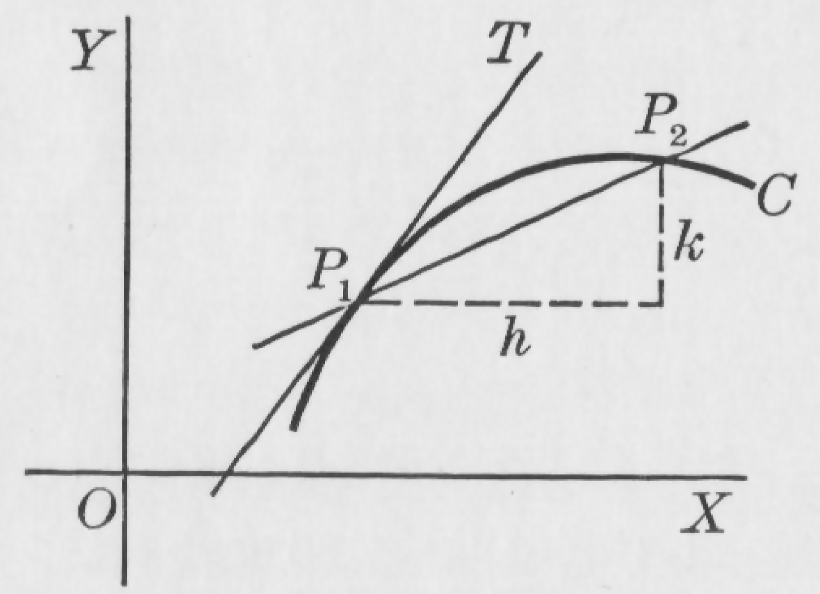


FIG. 99

To find an equation of the tangent line at a given point $P_1(x_1, y_1)$ on a curve whose equation is given, we need to find the slope m of the tangent line. This is obviously the limit of the slope of the secant line P_1P_2. Let the coördinates of P_2 be $(x_1 + h, y_1 + k)$. Then the slope of P_1P_2 is k/h, and hence

$$\boldsymbol{m} = \textbf{limit of } \frac{\boldsymbol{k}}{\boldsymbol{h}} \text{ as } P_2 \text{ approaches } P_1$$

$$= \text{limit of } \frac{k}{h} \text{ as } h \text{ and } k \text{ approach zero.}$$

The method of calculating this limit is illustrated in the following examples:*

* The student must not make the mistake of thinking that "the limit of k/h is $0/0 = 1$." The limit depends on the relative magnitudes of k and h as they approach zero. Thus if, for example, $k = ah$ where a is any constant, then $k/h = a$, and the limit of $k/h = a$.

Example 1. — Find the equation of the tangent to the parabola

$$y^2 = 2px$$

at a point $P_1(x_1, y_1)$ on the curve.

Solution. — The equation must be satisfied by (x_1, y_1); thus

$$(1) \qquad y_1^2 = 2px_1.$$

If $P_2(x_1 + h, y_1 + k)$ lies on the curve, then

$$(y_1 + k)^2 = 2p(x_1 + h),$$

or

$$(2) \qquad y_1^2 + 2ky_1 + k^2 = 2px_1 + 2ph.$$

Subtracting (1) from (2), we get

$$2ky_1 + k^2 = 2ph,$$

whence

$$(3) \qquad \frac{k}{h} = \frac{2p}{2y_1 + k}.$$

As h and k approach zero, the numerator of the right member has the constant value $2p$, and the denominator approaches $2y_1$; hence the fraction approaches p/y_1 as a limit. The slope m of the tangent line at $P_1(x_1, y_1)$ is therefore

$$(4) \qquad m = \frac{p}{y_1}.$$

Since the line passes through $P(x_1, y_1)$ its equation is

$$(5) \qquad y - y_1 = \frac{p}{y_1}(x - x_1).$$

This equation may be simplified. Clearing of fractions we have

$$yy_1 - y_1^2 = px - px_1;$$

adding equation (1) we obtain the simple formula *

$$(6) \qquad yy_1 = p(x + x_1).$$

* The derivation of this formula breaks down when $y_1 = 0$, but we verify readily that formula (6) is still true in this case. A similar remark will apply to all of the formulas of the Theorem on page 201. The tangent line is parallel to the y-axis, and the equation of the tangent has the form $x = a$ in all these cases.

Example 2. — Find the equation of the tangent to the ellipse

$$b^2x^2 + a^2y^2 = a^2b^2 \tag{7}$$

at a point $P_1(x_1, y_1)$ on the curve.

Solution. — Since P_1 is on the curve we have

$$b^2x_1^2 + a^2y_1^2 = a^2b^2. \tag{8}$$

The point $P_2(x_1 + h, y_1 + k)$ must also satisfy equation (7); hence

$$b^2(x_1 + h)^2 + a^2(y_1 + k)^2 = a^2b^2,$$

or

$$b^2x_1^2 + 2b^2hx_1 + b^2h^2 + a^2y_1^2 + 2a^2ky_1 + a^2k^2 = a^2b^2. \tag{9}$$

Subtracting (8) from (9) we get

$$2b^2hx_1 + b^2h^2 + 2a^2ky_1 + a^2k^2 = 0,$$

and hence

$$(2a^2y_1 + a^2k)k = -(2b^2x_1 + b^2h)h;$$

thus we find

$$\frac{k}{h} = -\frac{2b^2x_1 + b^2h}{2a^2y_1 + a^2k} = \text{slope of secant } P_1P_2. \tag{10}$$

As P_2 approaches P_1, h and k approach zero and the right member approaches $-b^2x_1/a^2y_1$ as a limit. Hence the slope m of the tangent line at P_1 is

$$m = -\frac{b^2x_1}{a^2y_1}.$$

The equation of the tangent line is therefore

$$y - y_1 = -\frac{b^2x_1}{a^2y_1}(x - x_1),$$

or

$$a^2y_1y - a^2y_1^2 = -b^2x_1x + b^2x_1^2.$$

The equation simplifies by transposing terms and then adding equation (8); we thus obtain

$$b^2x_1x + a^2y_1y = a^2b^2. \tag{11}$$

By the method of the examples the following theorem is proved.*

* In the seventeenth century methods were developed by Sir Isaac Newton and G. W. Leibnitz by means of which the slope of a tangent to a curve is very quickly found. These methods and their applications are the subject matter of differential calculus.

Theorem. *The equation of the tangent at the point of contact* $P_1(x_1, y_1)$

for the circle $x^2 + y^2 = a^2$ *is* $\mathbf{x_1x + y_1y = a^2}$;
for the parabola $y^2 = 2px$ *is* $\mathbf{y_1y = p(x + x_1)}$;

for the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ *is* $\mathbf{\frac{x_1x}{a^2} + \frac{y_1y}{b^2} = 1}$;

for the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ *is* $\mathbf{\frac{x_1x}{a^2} - \frac{y_1y}{b^2} = 1}$;

and for the general second degree curve

$$ax^2 + 2bxy + cy^2 + 2dx + 2ey + f = 0$$

it is $\mathbf{ax_1x + b(x_1y + y_1x) + cy_1y + d(x + x_1) + e(y + y_1) + f = 0}$.

EXERCISES

Use the method of the preceding examples (not merely the formulas of the preceding Theorem), to find the equation of the tangent at the point of contact $P_1(x_1, y_1)$ *for each of the curves whose equations follow.*

1. $x^2 = 2py$.
2. $x^2 + y^2 = a^2$.
3. $x^2 - y^2 = a^2$.
4. $2xy = a^2$.
5. $x^2 + y^2 = 2ax$.
6. $b^2x^2 - a^2y^2 = a^2b^2$.
7. $ax^2 + 2bxy + cy^2 + f = 0$.
8. $x^2 + y^2 + 2dx + 2ey + f = 0$.
9. $ax^2 + 2bxy + cy^2 + 2dx + 2ey + f = 0$.
10. $y^2 = x^3$ (*Semi-cubical parabola*).
11. $x^3 + y^3 = 3axy$ (*Folium*).
12. $x^2y + 4a^2y = 8a^3$ (*Witch*).
13. $x^3 + xy^2 + ax^2 = ay^2$ (*Strophoid*).

Use the theorem of § 91 *to find the equation of the tangent to the curve at the point of contact* P_1, *and draw the curve and the tangent line for each of the following curves and points.*

14. $y^2 = 8x$, $P_1(2, 4)$.
15. $y^2 = 16x$, $P_1(1, -4)$.
16. $x^2 + y^2 = 25$, $P_1(-3, 4)$.
17. $x^2 + y^2 = 53$, $P_1(2, 7)$.
18. $9x^2 + 4y^2 = 144$, $P_1(2, \sqrt{27})$.
19. $x^2 + 4y^2 = 52$, $P_1(6, 2)$.

20. $x^2 - y^2 = 60$, $P_1(8, 2)$.

21. $4x^2 - 9y^2 = 144$, $P_1(-9, \sqrt{20})$.

22. $x^2 + y^2 + 6x = 0$, $P_1(-4, \sqrt{8})$.

23. $2xy - 81 = 0$, at a point P_1 where $x_1 = 9$.

24. $x^2 + 4y^2 + 16x = 4$, at the point P_1 where $x_1 = -6$ and y_1 is positive.

25. Show analytically that the tangent line $x_1x + y_1y = a^2$ to the circle $x^2 + y^2 = a^2$ at the point of contact $P_1(x_1, y_1)$, is perpendicular to the radius to P_1.

26. Find equations of the lines passing through the point $R(3, 8)$ which are tangent to the parabola $y^2 = 16x$.

Hint. Let $P_1(x_1, y_1)$ be an unknown point of contact. Then the tangent is

$$y_1y = 8(x + x_1). \tag{1}$$

Since it passes through $R(3, 8)$ we must have

$$8y_1 = 8(3 + x_1). \tag{2}$$

Since P_1 lies on the parabola we must have

$$y_1^2 = 16x_1. \tag{3}$$

Solve (2) and (3) and substitute in (1).

27. Find equations of the lines which pass through the point $R(6, \sqrt{3})$ and are tangent to the ellipse $9x^2 + 4y^2 = 144$. Draw the curve and the lines.

28. Find equations of the lines which pass through the point $R(6\sqrt{5}, 9)$ and are tangent to the hyperbola $4x^2 - 9y^2 = 36$.

29. Prove that the tangents at the ends of the latus rectum of a parabola are perpendicular to each other.

30. Prove that if an ellipse and a hyperbola have the same foci, the tangents at a point of intersection are perpendicular to each other.

92. The normal to a curve. Let P_1 be a point on a curve C. The line through P_1 which is perpendicular to the tangent to the curve at P_1 is called the **normal** to the curve at P_1.

The slope of the normal is the negative reciprocal of the slope of the tangent, and hence the equation of the normal is quickly found from that of the tangent. Formulas for normals to conics are given in the following theorem.

Theorem. *The equation of the normal at the point* $P_1(x_1, y_1)$

for the circle $x^2 + y^2 = a^2$ *is* $y_1x - x_1y = 0$;
for the parabola $y^2 = 2px$ *is* $y_1x + py = x_1y_1 + py_1$;
for the ellipse $b^2x^2 + a^2y^2 = a^2b^2$ *is* $a^2y_1x - b^2x_1y = (a^2 - b^2)x_1y_1$;
for the hyperbola $b^2x^2 - a^2y^2 = a^2b^2$ *is* $a^2y_1x + b^2x_1y = (a^2 + b^2)x_1y_1$.

93. Lengths of tangent and normal. Subtangent and subnormal. Suppose that the tangent to a curve C at the point of contact P_1 cuts the x-axis at T, and that the normal cuts the x-axis at N. Let M_1 be the projection of P_1 on the x-axis. Then by definition

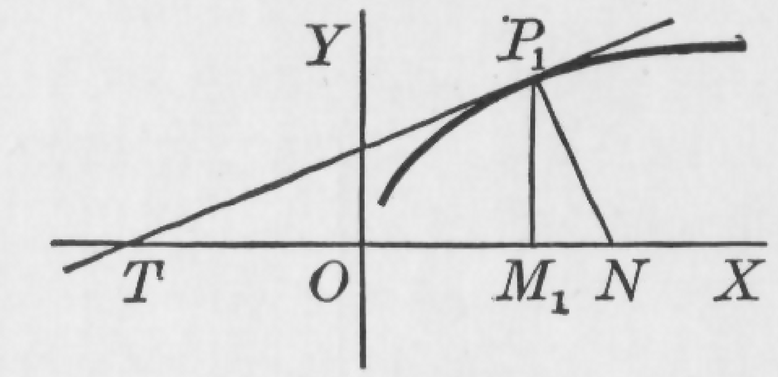


FIG. 100

$$\begin{aligned} \overline{TP_1} &= \textit{length of tangent at } P_1; \\ \overline{NP_1} &= \textit{length of normal at } P_1; \\ TM_1 &= \textit{subtangent at } P_1; \\ NM_1 &= \textit{subnormal at } P_1. \end{aligned} \tag{1}$$

The first two are always positive; the last two are directed lengths. In the figure, TM_1 is positive and NM_1 is negative.

These quantities are readily expressed in terms of the slope m of the tangent and the coördinates (x_1, y_1) of P_1. Thus

$$m = \frac{M_1P_1}{TM_1}, \qquad -\frac{1}{m} = \frac{M_1P_1}{NM_1},$$

and hence, since $M_1P_1 = y_1$,

$$TM_1 = \frac{y_1}{m} \qquad NM_1 = -my_1. \tag{2}$$

We then have, by the theorem of Pythagoras,

$$\overline{TP_1} = \sqrt{y_1^2 + \left(\frac{y_1}{m}\right)^2}, \qquad \overline{NP_1} = \sqrt{y_1^2 + m^2y_1^2}. \tag{3}$$

Example 1. — Find the lengths of the tangent and normal, and find the subtangent and subnormal to the parabola

$$2y = x^2 + 3$$

at the point $P_1(3, 6)$.

Solution. — The equation of the tangent can be found by applying the last formula of the Theorem of § 91. Writing the equation

$$x^2 - 2y + 3 = 0,$$

we have $a = 1$, $b = c = d = 0$, $e = -1$, $f = 3$. Hence for the point where $x_1 = 3$, $y_1 = 6$, the equation of the tangent is

$$3x - (y + 6) + 3 = 0.$$

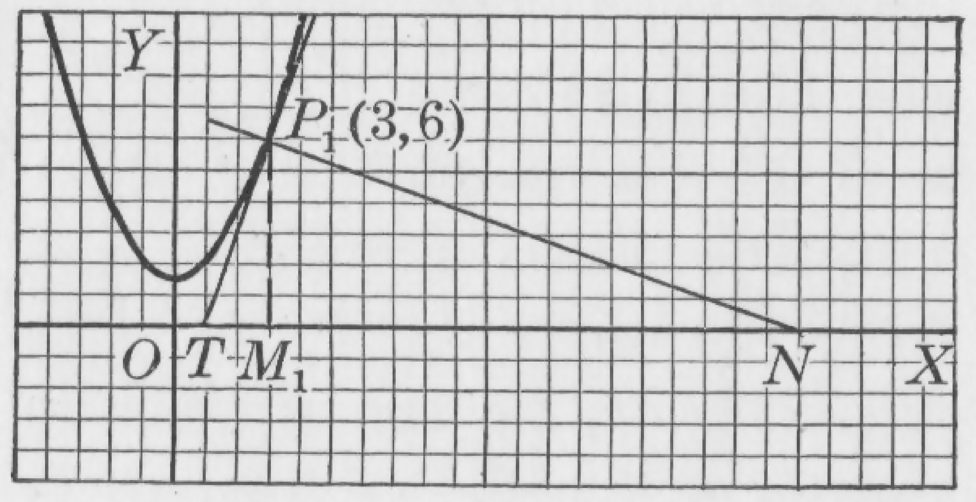


FIG. 101

The slope of this line is $m = 3$. Hence

$$\text{the subtangent} = \frac{y_1}{m} = 2,$$
$$\text{the subnormal} = -my_1 = -18,$$
$$\text{the length of tangent} = \sqrt{6^2 + 2^2} = 2\sqrt{10},$$
$$\text{the length of normal} = \sqrt{6^2 + 18^2} = 6\sqrt{10}.$$

Example 2. — Prove that the length of the subnormal of the parabola $y^2 = 2px$ is p, thus being independent of the point of contact.

Solution. — The equation of the tangent at (x_1, y_1) is

$$y_1y = p(x + x_1).$$

Hence $m = p/y_1$, and $NM_1 = -my_1 = -p$. The length of NM_1 is therefore p.

EXERCISES

Find the equation of the normal, find the lengths of the tangent and normal, and find the subtangent and subnormal to each of the following curves at the point P₁ *specified.*

1. $x^2 + y^2 = 25$, $P_1\,(3, 4)$. **2.** $x^2 + y^2 = 29$, $P_1(5, -2)$.

3. $y^2 = 16x$, $P_1(9, 12)$. **4.** $y^2 = 12x$, $P_1(3, -6)$.

5. $x^2 + 4y^2 = 25$, $P_1(3, 2)$. **6.** $9x^2 + 4y^2 = 100$, $P_1(2, -4)$.

7. $x^2 - 16y^2 = 36$, $P_1(10, 2)$. **8.** $9x^2 - y^2 = 81$, $P_1(-5, 12)$.

9. $x^2 + y^2 + 6x = 52$, $P_1(3, 5)$.

10. $x^2 + 4y^2 - 8y = 37$, $P_1(-5, 3)$.

11. $x^2 + 4x + 8y = 13$, $P_1(-3, 2)$.

12. $x^2 - 2xy + y^2 = 1$, $P_1(5, 4)$.

13. $y^2 = x^3$, $P_1(4, -8)$.

14. $x^3 + y^3 = 3axy$, $P_1(x_1, y_1)$ (*Folium*).

15. $x^2y + 4a^2y = 8a^3$, $P_1(2a, a)$ (*Witch*).

16. $x^3 + xy^2 + ax^2 = ay^2$, $P_1(x_1, y_1)$ (*Strophoid*).

17. $10y = x^3 + 7x^2 - 24x - 30$, $x_1 = -6$.

18. Show that the subtangents to the parabola $y^2 = 2px$ are bisected by the vertex. Show how to draw a tangent, using this property.

19. Derive the formula for the equation of the normal at $P_1(x_1, y_1)$ to

(*a*) the parabola $y^2 = 2px$;
(*b*) the ellipse $b^2x^2 + a^2y^2 = a^2b^2$;
(*c*) the hyperbola $b^2x^2 - a^2y^2 = a^2b^2$.

20. Show that the area of the triangle bounded by a tangent to the curve $2xy = a^2$ and the coördinate axes is constant; that is, it does not depend on the point of contact $P_1(x_1, y_1)$.

21. Prove that the area of the triangle bounded by a tangent to any hyperbola and its asymptotes is independent of the point of contact.

22. Prove that normals to a parabola at the ends of a chord which passes through the focus are perpendicular to each other.

23. Draw the ellipse $b^2x^2 + a^2y^2 = a^2b^2$ and its major auxiliary circle $x^2 + y^2 = a^2$. Prove the equality of the subtangents to the two curves at points whose abscissas are the same, $x = x_1$, thus showing that tangents at corresponding points on the two curves meet on the major axis of the ellipse.

24. Derive an equation of the normal at a point $P_1(x_1, y_1)$ to the curve having the equation

$$ax^2 + 2bxy + cy^2 + 2dx + 2ey + f = 0,$$

the point $P_1(x_1, y_1)$ being on the curve. Also find an expression for the subnormal.

94. Equation of tangent in terms of slope. If the equation of a curve is given, how can we find the equation of a tangent line which has a given slope? A general method is as follows:

Let the slope of the tangent be m. Then the equation of the tangent has the form

(1) $$y = mx + k.$$

In general the line (1) cuts the curve in two or more distinct points, as illustrated in Figure 102. If we determine k so that two of these points coincide, the line (1) is in general a tangent line.

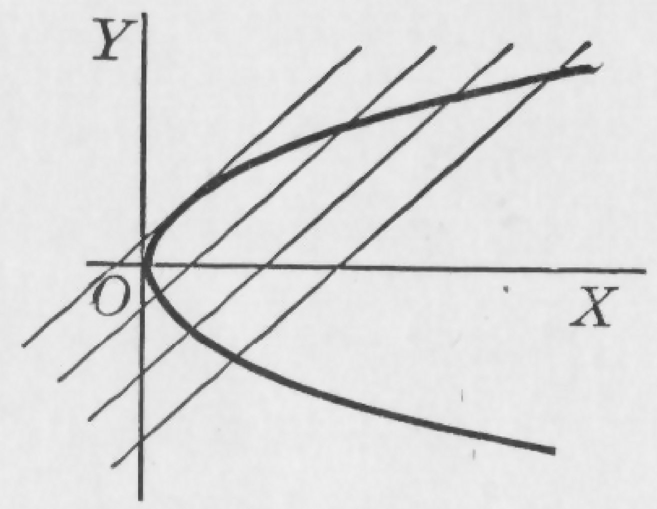


FIG. 102

Example 1. — Find tangents to the parabola $y^2 = 12x$ which have the slope 2.

Solution. — The points of intersection of the parabola and the line

$$y = 2x + k$$

are found by solving the equations simultaneously. At such points

$$(2x + k)^2 = 12x,$$
$$4x^2 + (4k - 12)x + k^2 = 0.$$

In general there are two values of x satisfying this equation. The two will coincide if the discriminant is zero (see page 1); that is, if

$$(4k - 12)^2 - 16k^2 = 0, \quad \text{or} \quad k = 3/2.$$

Hence there is only one tangent line having the given slope; its equation is $y = 2x + 3/2$.

Example 2. — Find the equations of the lines of slope m which are tangent to the ellipse

(1) $$b^2x^2 + a^2y^2 = a^2b^2.$$

Solution. — The equations have the form

(2) $$y = mx + k.$$

At points of intersection of (2) and (1) we have

$$b^2x^2 + a^2(mx + k)^2 = a^2b^2,$$

or

$$(b^2 + a^2m^2)x^2 + 2a^2mkx + a^2k^2 - a^2b^2 = 0.$$

The line (2) cuts the ellipse (1) in two coincident points if and only if

$$4a^4m^2k^2 - 4(b^2 + a^2m^2)(a^2k^2 - a^2b^2) = 0,$$

whence, on solving for k, we have $k = \pm\sqrt{a^2m^2 + b^2}$. Thus there are two lines of slope m which are tangent to the ellipse:

$$y = mx + \sqrt{a^2m^2 + b^2}; \qquad y = mx - \sqrt{a^2m^2 + b^2}.$$

EXERCISES

Find equations of lines with the given slope which are tangent to each of the following curves. Draw the curve and tangents.

1. $x^2 + y^2 = 25,\quad m = -3/4.$
2. $x^2 + y^2 + 4x = 32,\quad m = -1.$
3. $y^2 = 6(x - 2),\quad m = -2.$
4. $x^2 = 8(y + 4),\quad m = -3.$
5. $36x^2 + y^2 = 144,\quad m = 1.$
6. $9x^2 + 4y^2 = 64,\quad m = -2.$
7. $x^2 - y^2 = 81,\quad m = 2.$
8. $4x^2 - 9y^2 + 36 = 0, m = 3.$

Show that equations of the tangents of slope m *are as given for each of the curves of Exercises* 9–11.

9. For the circle $x^2 + y^2 = a^2$, $\quad y = mx \pm a\sqrt{m^2 + 1}$.

10. For the parabola $y^2 = 2px$, $\quad y = mx + \dfrac{p}{2m}\cdot$

11. For the hyperbola $b^2x^2 - a^2y^2 = a^2b^2$, $y = mx \pm \sqrt{a^2m^2 - b^2}$.

12. Two mutually perpendicular tangent lines to the parabola $y^2 = 2px$ intersect at $P(x, y)$. Prove that the locus of P is the directrix $x + \dfrac{p}{2} = 0$.

13. Two mutually perpendicular tangent lines to the ellipse $b^2x^2 + a^2y^2 = a^2b^2$ intersect at $P(x, y)$. Prove that the locus of P is the circle $x^2 + y^2 = a^2 + b^2$. Illustrate with a figure, using $a = 12$, $b = 5$. This circle is called the **director circle** of the ellipse.

14. If, in Exercise 13, the ellipse is replaced by the hyperbola $b^2x^2 - a^2y^2 = a^2b^2$, what is the locus of P? Illustrate with a figure, using $a = 10$, $b = 8$.

95. Reflection property of the parabola. At any point P_1 of a parabola draw a line P_1F to the focus, a line P_1A parallel to the axis, and the internal normal P_1N to the curve. We shall prove that the normal bisects the angle between the other two lines.

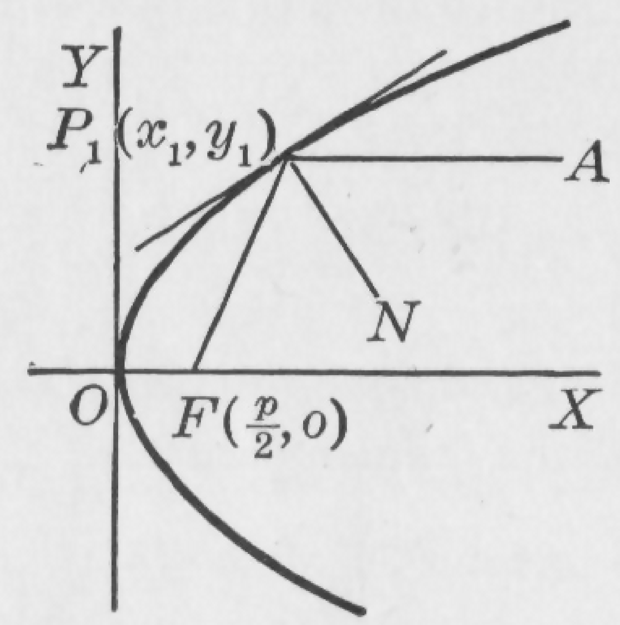


FIG. 103

Choosing axes of coördinates properly, we have for the equation of the curve

$$y^2 = 2px.$$

The slope of the normal P_1N at $P_1(x_1, y_1)$ is found from its equation in § 92; it is $-y_1/p$. The slope of the focal radius FP_1 is $y_1/(x_1 - p/2)$. Hence, by (1) page 40,

$$\tan \angle FP_1N = \frac{-\dfrac{y_1}{p} - \dfrac{y_1}{x_1 - p/2}}{1 - \dfrac{y_1^2}{p(x_1 - p/2)}},$$

$$= \frac{-y_1(x_1 + p/2)}{px_1 - p^2/2 - y_1^2}.$$

Since $y_1^2 = 2px_1$, this reduces to

$$\tan \angle FP_1N = \frac{y_1}{p}.$$

Since P_1A is parallel to the x-axis,

$$\tan \angle NP_1A = -\text{ slope } P_1N$$

$$= \frac{y_1}{p}.$$

Hence

$$\angle FP_1N = \angle NP_1A.$$

Applications of this property of a parabola are explained in § 45, page 115.

96. Reflection property of an ellipse. Let $P(x, y)$ be a point on the ellipse

$$b^2x^2 + a^2y^2 = a^2b^2.$$

We shall prove that the normal PN to the ellipse bisects the angle between the focal radii FP and $F'P$ (Fig. 104).

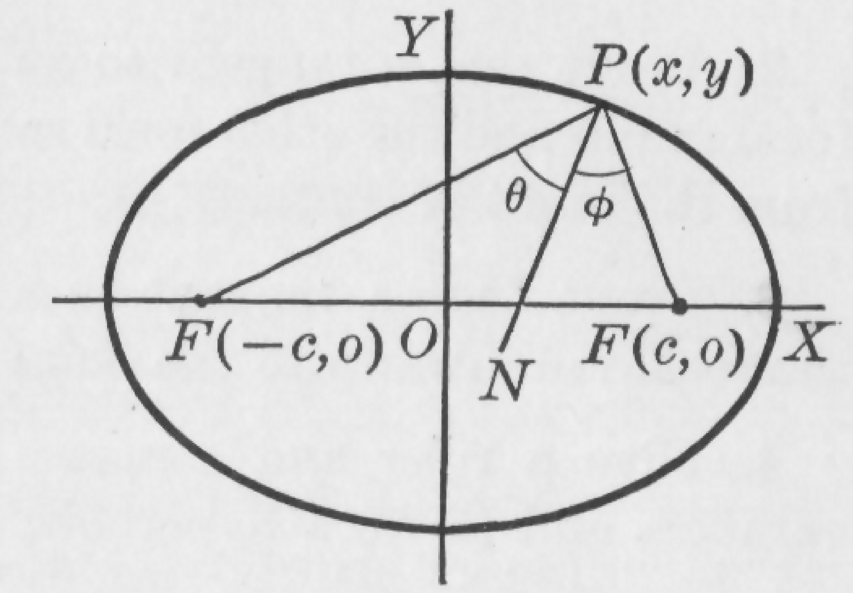


FIG. 104

It will suffice to show, in the notation of the figure, that

$$\tan \theta = \tan \phi.$$

We have

$$\text{slope of } F'P = \frac{y}{x + c}, \quad \text{slope of } FP = \frac{y}{x - c},$$

and, from § 92,

$$\text{slope of } NP = \frac{a^2y}{b^2x}.$$

Hence

$$\tan \theta = \frac{\dfrac{a^2y}{b^2x} - \dfrac{y}{x + c}}{1 + \dfrac{a^2y^2}{b^2x(x + c)}} = \frac{a^2xy + a^2cy - b^2xy}{b^2x^2 + b^2cx + a^2y^2}$$

$$= \frac{c^2xy + a^2cy}{a^2b^2 + b^2cx} = \frac{cy}{b^2}.$$

$$\tan \phi = \frac{\dfrac{y}{x - c} - \dfrac{a^2y}{b^2x}}{1 + \dfrac{a^2y^2}{b^2x(x - c)}} = \frac{b^2xy - a^2xy + a^2cy}{b^2x^2 - b^2cx + a^2y^2}$$

$$= \frac{-c^2xy + a^2cy}{a^2b^2 - b^2cx} = \frac{cy}{b^2}.$$

Thus $\theta = \phi$, since $\tan \theta = \tan \phi$.

As stated in § 50, page 124, this theorem concerning an ellipse provides an explanation of the so-called "whispering galleries."

EXERCISES

1. Prove that a tangent to a parabola bisects the external angle between a line parallel to the axis and the focal radius, these lines being drawn from the point of contact.

2. Prove that a tangent to an ellipse bisects the angle between one focal radius and the other focal radius produced, these lines being drawn from the point of contact.

3. Prove that a tangent to a hyperbola bisects the angle between the focal radii drawn to the point of contact.

4. Give a ruler and compass construction of a tangent (*a*) to a parabola and (*b*) to a hyperbola, the foci being given.

5. Prove that tangents to a parabola from a point on the directrix are perpendicular to each other.

6. Prove that the line drawn from a focus of an ellipse perpendicular to a tangent, and the line passing through the center and the point of contact, intersect on a directrix of the ellipse. Is the corresponding statement true for a hyperbola?

7. Prove that the product of the distances from the foci to a tangent of an ellipse is the same for all tangents. Is the corresponding statement true for a hyperbola?

CHAPTER XI

DIAMETERS, POLES AND POLARS

97. Diameters of an ellipse. The locus of the mid-points of all chords parallel to a given chord of a conic section is called a **diameter** of the conic.* Thus the axes of an ellipse are diameters, since each bisects all chords parallel to the other.

To determine other diameters of an ellipse, let a chord of slope m meet the ellipse

$$b^2x^2 + a^2y^2 = a^2b^2$$

in the points $P_1(x_1, y_1)$ and $P_2(x_1 + h, y_1 + k)$. Let $P(x, y)$ be the mid-point of the chord P_1P_2; then

$$x = \frac{2x_1 + h}{2}, \qquad y = \frac{2y_1 + k}{2}.$$

Now by equation (10), page 200, we have

$$m = \frac{k}{h} = -\frac{b^2(2x_1 + h)}{a^2(2y_1 + k)};$$

hence

$$m = -\frac{b^2x}{a^2y},$$

or

$$y = -\frac{b^2}{a^2m}x. \tag{1}$$

Thus the diameter lies on the straight line (1); it consists of the segment of that line which is interior to the ellipse. Hence we have the following theorem.

---

* Note that this is a property of a diameter of a circle.

A diameter of the ellipse $b^2x^2 + a^2y^2 = a^2b^2$ *is a segment of a straight line* (called a **diametral line**) *which passes through the center of the ellipse; its slope* m′ *is related to the slope* m *of the chords it bisects by the formula*

$$(2) \qquad m' = -\frac{b^2}{a^2m}, \quad or \quad mm' = -\frac{b^2}{a^2}.$$

Let AB and CD be two chords through the center of an ellipse such that their slopes are related by the formula

$$mm' = -\frac{b^2}{a^2}.$$

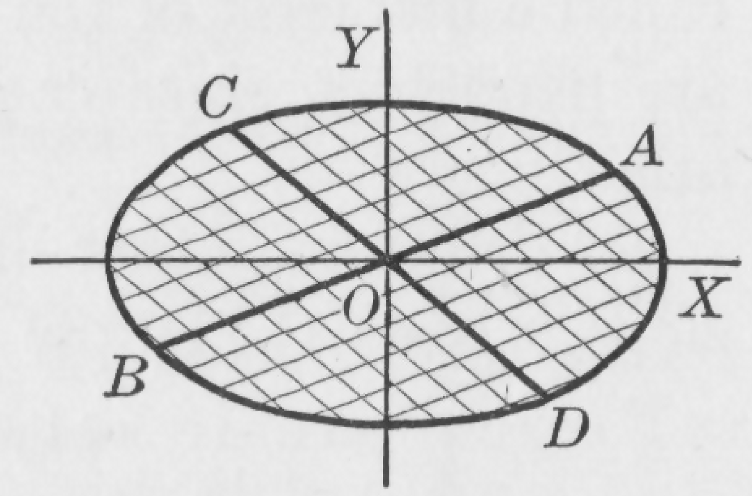


FIG. 105

It follows from the preceding theorem and from the definition of a diameter that each of these chords is a diameter bisecting all chords parallel to the other. Such diameters are called **conjugate diameters** of the ellipse. To each diameter there corresponds one and only one conjugate diameter.

The axes of the ellipse are an exceptional pair of conjugate diameters which do not verify formula (2).

EXERCISES

1. Draw accurately * an ellipse whose major and minor axes are 10 cm. and 6 cm. long. Draw diameters of inclination 10°, 20°, 30°, 40°, 50°, 60°, 70°, 80° and draw the conjugate of each. Draw four chords of inclination 20° and verify by measurement the statement that they are bisected by the appropriate diameter.

2. Find a pair of conjugate diameters which make equal angles with the major axis of the ellipse $b^2x^2 + a^2y^2 = a^2b^2$.

3. Prove that if two conjugate diameters other than the axes of an ellipse are mutually perpendicular then the ellipse is a circle.

* For the problems of this chapter it is desirable that a very accurate figure be drawn from which copies may be made. A piece of cardboard cut carefully would be helpful.

4. Prove that if one diameter of the ellipse $b^2x^2 + a^2y^2 = a^2b^2$ lies in the first and third quadrants then its conjugate lies in the second and fourth quadrants.

5. If the slopes of two conjugate diameters of the ellipse $b^2x^2 + a^2y^2 = a^2b^2$ are 10/9 and − 4/5, what is the eccentricity of the ellipse ? And what are the equations of the conjugate diametral lines which make equal angles with the minor axis?

6. Prove analytically that the tangents to an ellipse at the ends of a diameter are parallel to the conjugate diameter.

7. Show that the equation of the diametral line which bisects chords of slope m for the ellipse

$$\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$$

is

$$y - k = -\frac{b^2}{a^2m}(x - h).$$

8. Find the equation of the diametral line which bisects chords of slope 2 for the ellipse

$$x^2 + 4y^2 - 2x + 8y = 11.$$

98. Diameters of a hyperbola. The discussion of diameters of a hyperbola runs closely parallel to that of the preceding section. Since the standard equation of a hyperbola is obtained from that of an ellipse by changing the sign of b^2, we have at once the theorem:

A diameter of the hyperbola

$$(1) \qquad b^2x^2 - a^2y^2 = a^2b^2$$

is all or a portion of a straight line (called a **diametral line**) *which passes through the center of the hyperbola; its slope* m′ *is related to the slope* m *of the chords which it bisects by the formula*

$$(2) \qquad mm' = \frac{b^2}{a^2}.$$

Two diameters whose slopes are related by formula (2) are called **conjugate diameters**. Each bisects all chords parallel to the other. The axes of (1) are on conjugate diametral

lines which do not verify formula (2). A pair of conjugate diameters is shown by heavy lines in Figure 106.

Consider now the hyperbola

(3) $\quad - b^2x^2 + a^2y^2 = a^2b^2,$

conjugate to the hyperbola (1). Its equation is obtained from the preceding by changing the signs of both a^2 and b^2. Hence facts concerning its diameters may be obtained by the same change of signs.

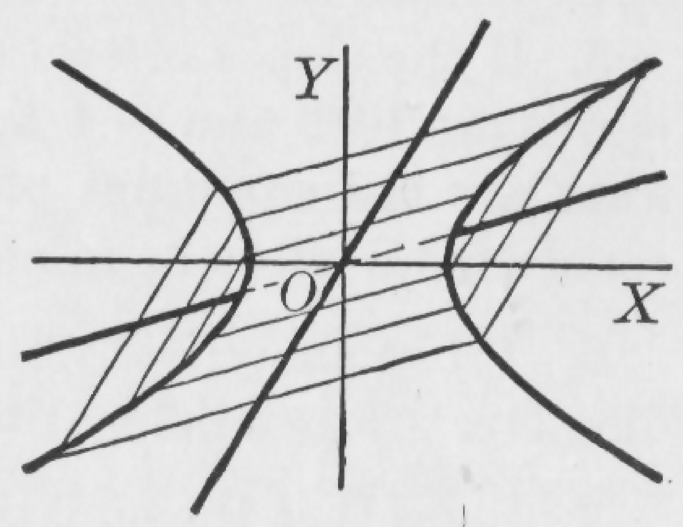


FIG. 106

Since equation (2) remains unaltered by this change, it follows that *conjugate diametral lines of the hyperbola* (1) *are conjugate diametral lines of its conjugate hyperbola* (3).

EXERCISES

1. Draw accurately the hyperbola $\frac{x^2}{25} - \frac{y^2}{9} = 1$. Draw diameters of inclination 10°, 20°, 30°, and their conjugates. Draw four chords of inclination 20° and verify by measurement the statement that they are bisected by the appropriate diameter.

2. Draw carefully a system of chords of slope 2 for the hyperbola $16x^2 - 25y^2 = 400$ and its conjugate. Draw the diametral line which bisects all of these chords and find its equation.

3. Prove that if a diameter of the hyperbola $b^2x^2 - a^2y^2 = a^2b^2$ lies in the first and third quadrants, then its conjugate diameter does also.

4. Prove that any pair of conjugate diameters of the equilateral hyperbola $x^2 - y^2 = a^2$ makes equal angles with an asymptote.

5. Prove that conjugate diametral lines of the hyperbola

$$b^2x^2 - a^2y^2 = a^2b^2$$

are conjugate diametral lines of the hyperbola

$$b^2x^2 - a^2y^2 = ka^2b^2.$$

6. Prove that the asymptotes of the hyperbola

$$b^2x^2 - a^2y^2 = a^2b^2$$

are conjugate diametral lines of the ellipse

$$b^2x^2 + a^2y^2 = a^2b^2.$$

99. Diameters of a parabola. Let a chord of slope m intersect the parabola

$$y^2 = 2px$$

in the points $P_1(x_1, y_1)$ and $P_2(x_1 + h, y_1 + k)$. Its midpoint $P(x, y)$ is such that

$$x = \frac{2x_1 + h}{2}, \quad y = \frac{2y_1 + k}{2}.$$

Since P_1 and P_2 lie on the curve we have

$$y_1^2 = 2px_1,$$
$$(y_1 + k)^2 = 2p(x_1 + h),$$

whence, by subtracting the former from the latter equation,

$$(2y_1 + k)k = 2ph$$

or

$$\frac{k}{h} = \frac{2p}{2y_1 + k} = \frac{p}{y}.$$

Since $m = k/h$, it follows that

(1) $$y = \frac{p}{m}.$$

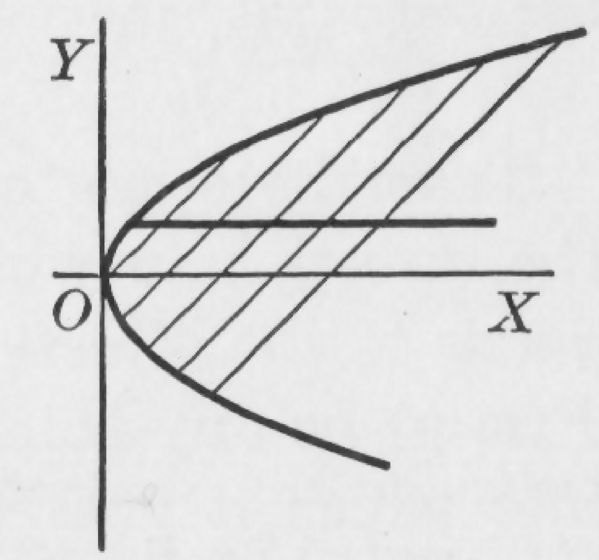


FIG. 107

Hence *a diameter of a parabola is a portion of a line parallel to the axis of the parabola; its distance from the axis is* p/m, *where* m *is the slope of the chords it bisects. The axis of the parabola is also a diameter, which bisects chords perpendicular to it.*

EXERCISES

1. Derive analytically the equation satisfied by points on the diameter of the parabola $x^2 = 2py$ which bisects chords of slope m.

Draw carefully each of the following parabolas and draw four chords of the given slope. Write the equation of the diametral line which bisects the chords, plot it, and observe that the chords are bisected.

2. $y^2 = 16x, \quad m = \frac{1}{2}.$ **3.** $y^2 = 8x, \quad m = 2.$

4. $y^2 = -16x, \quad m = 2.$ **5.** $y^2 = -8x, \quad m = \frac{1}{2}.$

6. $x^2 = 16y, \quad m = \frac{1}{4}.$ **7.** $x^2 = -8y, \quad m = -2.$

100. A physical interpretation of conjugate diameters. Suppose that a circle and two mutually perpendicular diameters are drawn on a wooden disc. Now suppose that the disc is subjected to pressure such that in a certain direction in the plane of the circle all parts are uniformly compressed while in the direction at right angles to this one there is neither compression nor expansion. Let us show that the circle goes into an ellipse and that the perpendicular diameters go into conjugate diameters.

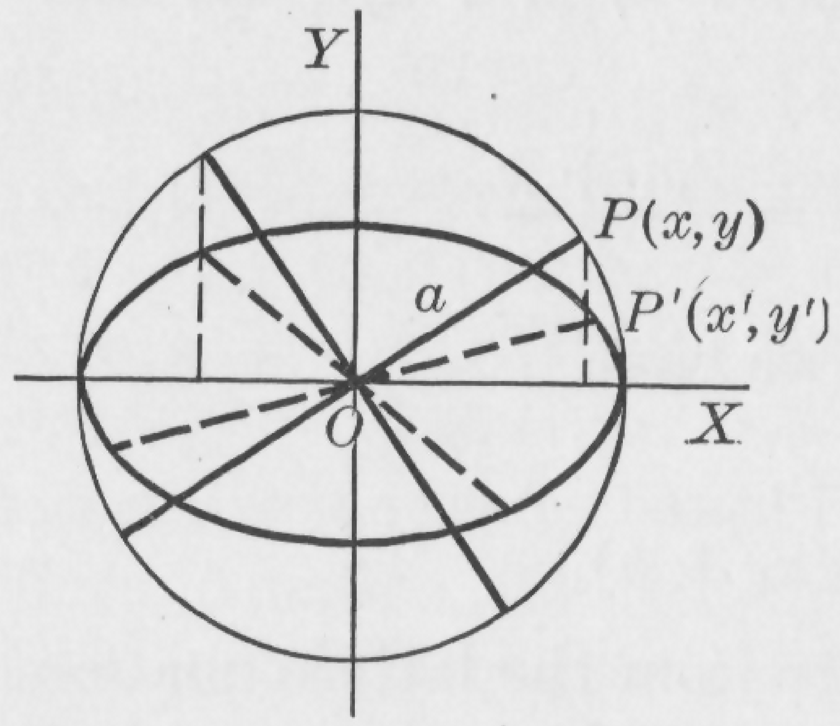


FIG. 108

Take the origin at the center of the circle, the x-axis in the direction in which there was no compression, and the y-axis in the direction in which compression occurred. Let $P(x, y)$ be any point on the circle, and $P'(x', y')$ the point into which it goes through the compression. Then under our hypotheses

$$x = x', \quad y = ky',$$

where k is a constant greater than unity. Since

$$x^2 + y^2 = a^2,$$

where a is the radius of the circle, we have

$$x'^2 + k^2y'^2 = a^2.$$

This is the equation of an ellipse in which

$$b^2 = a^2/k^2.$$

Let the equations of the mutually perpendicular diametral lines of the circle be

$$y = mx \quad \text{and} \quad y = -\frac{x}{m}.$$

These lines go into

$$ky' = mx' \quad \text{and} \quad ky' = -\frac{x'}{m}.$$

The latter are two straight lines which pass through the center of the ellipse, and the product of their slopes is $-1/k^2$, which equals $-b^2/a^2$. Hence they are conjugate diametral lines. The truth of the statement which we were to prove has thus been established.

101. Polar of a point. Let $P_1(x_1, y_1)$ be a point exterior to the ellipse

$$(1) \qquad b^2x^2 + a^2y^2 = a^2b^2.$$

Through P_1 there pass two lines which are tangent to the ellipse; let $P_2(x_2, y_2)$ and $P_3(x_3, y_3)$ be their points of contact. The equation of one of these tangents is, by equation (11), page 200,

$$b^2x_2x + a^2y_2y = a^2b^2$$

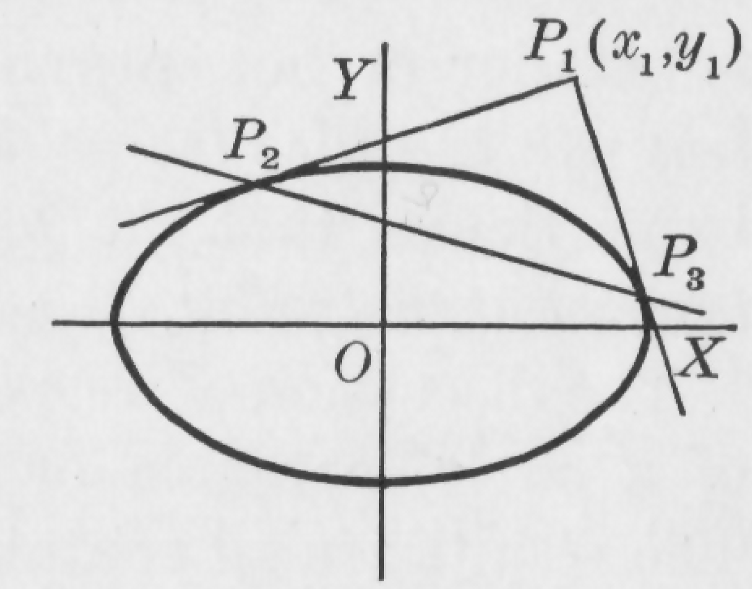


FIG. 109

and that of the other is

$$b^2x_3x + a^2y_3y = a^2b^2.$$

Since these lines go through $P_1(x_1, y_1)$ we have

$$(2) \qquad \begin{aligned} b^2x_2x_1 + a^2y_2y_1 &= a^2b^2, \\ b^2x_3x_1 + a^2y_3y_1 &= a^2b^2. \end{aligned}$$

Consider now the line

$$(3) \qquad b^2x_1x + a^2y_1y = a^2b^2.$$

We prove that it is the secant line through P_2 and P_3 by observing that the equation (3) is satisfied by both (x_2, y_2) and (x_3, y_3), as shown by equations (2), and that it is the equation of a straight line.

The straight line P_2P_3 through the points of contact of tangents from P_1 to the ellipse is called the **polar** of the point P_1 with respect to the ellipse. We have the theorem:

The polar of the point $P_1(x_1, y_1)$ *with respect to the ellipse*

$$b^2x^2 + a^2y^2 = a^2b^2$$

is the straight line

$$b^2x_1x + a^2y_1y = a^2b^2. \tag{3}$$

We note that this equation (3) is of precisely the same form as the equation of the tangent to the ellipse at a point $P_1(x_1, y_1)$ on the ellipse.

In the preceding discussion $P_1(x_1, y_1)$ was a point outside the ellipse. The formula (3) gives a real line whether P_1 is exterior or not, provided P_1 is not at the origin. *This line will be called the polar of* P_1 *in all cases.* Thus it occurs in particular that *the polar of a point on an ellipse is the tangent at that point.*

The discussion corresponding to the preceding for the case of a circle, parabola or hyperbola presents no difficulties. The results are all contained in the statement:

The equation of the polar of a point $P_1(x_1, y_1)$ *with respect to a conic is of precisely the same form as that of the equation of the tangent to the conic at a point* $P_1(x_1, y_1)$ *on the conic* (see page 201).

EXERCISES

1. Draw accurately an ellipse whose major axis is 10 cm. and whose minor axis is 6 cm. long, and write its standard equation. Find the equations of the polars with respect to this ellipse of the points

$$A(5, 3),\ B(10, -6),\ C(-4, -\tfrac{9}{5}),\ D(-4, 0),\ E(-2, 2).$$

Plot each polar and from the drawing verify when possible that the polar of a point passes through the points of contact of tangents to the ellipse from the point.

2. Draw the circle

$$x^2 + y^2 = 36$$

and find the equation of the polar with respect to the circle for each of the following points which has a polar:

$$A(6, 6),\ B(8, -2),\ C(-4, 2\sqrt{5}),\ D(-2, -2),\ E(0, 0).$$

Plot each polar and from the drawing verify when possible that the polar of a point passes through the points of contact of tangents to the circle from the point.

3. Draw accurately the hyperbola

$$\frac{x^2}{25} - \frac{y^2}{9} = 1,$$

and find the equations of the polars with respect to the hyperbola of the points

$$A(5, 3),\ B(10, -6),\ C(-10, 3\sqrt{3}),\ D(-4, 0),\ E(-2, -2).$$

Plot each polar and from the drawing verify when possible that the polar of a point passes through the points of contact of tangents to the hyperbola from the point.

4. Draw accurately the parabola

$$y^2 = 16x,$$

and find the equations of the polars with respect to the parabola of the points

$$A(0, 4),\ B(-2, 5),\ C(-4, -2),\ D(2, -2),\ E\left(\frac{25}{9}, \frac{20}{3}\right).$$

Plot each polar and from the drawing verify when possible that the polar of a point passes through the points of contact of tangents to the parabola from the point.

5. Let C be a circle, ellipse or hyperbola. Show that every point in the plane, except the center of C, has a polar with respect to C. Show that the center of C has no polar with respect to C.

6. Prove that the polar of the focus of a parabola with respect to the parabola is the directrix.

7. Prove that the polar of the focus of an ellipse with respect to the ellipse is the corresponding directrix.

8. Prove that with respect to a hyperbola the polar of a focus is the corresponding directrix.

9. Prove that the polar of any point P_1 with respect to a circle is perpendicular to the line joining P_1 to the center of the circle.

10. Let O be the center and a the radius of a circle. Prove that the product of the distance from O to any point P_1 (other than O) and the distance from O to the polar of P_1 with respect to the circle is a^2.

11. Use the results of Exercises 9 and 10 to discuss the motion of the polar of P_1 as P_1 approaches the center O along a radius. Likewise as P_1 describes a circle about O as center.

12. Prove that with respect to a hyperbola the polar of a point on an asymptote, other than the center, is a line parallel to the asymptote.

102. Pole of a line. Let C be a conic. If the line L is the polar of a point P_1 with respect to C, then the point P_1 is called a **pole** of the line L with respect to C.

If the conic C and the line L are given, we find a pole P_1 of L as follows.

Suppose that the conic is the ellipse or hyperbola

$$Ax^2 + By^2 = 1.$$

The equation of any line which is not a diametral line (or asymptote) can be written in the form

$$(1) \qquad ax + by = 1.$$

A point $P_1(x_1, y_1)$ is the pole of (1) with respect to the ellipse if and only if the line (1) and the line

$$(2) \qquad Ax_1x + By_1y = 1$$

coincide; that is, if $x_1 = a/A$, $y_1 = b/B$. *Thus the line* (1) *has the unique pole* $P_1(a/A, b/B)$.

The equation of a diametral line (or asymptote) can be written in the form

$$ax + by = 0.$$

This coincides with the polar (2) of a point $P_1(x_1, y_1)$ only if

$$Ax_1 : a = By_1 : b = 1 : 0.$$

Since there are no values of (x_1, y_1) satisfying this proportion, *a diametral line* (or asymptote) *has no pole.*

Consider next the case of the parabola

$$y^2 = 2px.$$

The equation of any line which is not a diametral line can be written

$$(3) \qquad x + By + C = 0.$$

The polar of $P_1(x_1, y_1)$ with respect to this parabola is the line

$$(4) \qquad yy_1 = px + px_1.$$

These lines (3) and (4) coincide if and only if

$$-p : 1 = y_1 : B = -px_1 : C,$$

or

$$x_1 = C, \qquad y_1 = -Bp.$$

Thus there is a *unique pole of the line* (3); *it is the point* $P_1(C, -Bp)$.

The equation of a diametral line of the parabola can be written in the form

$$y = b.$$

This line and (4) coincide only if

$$-p : 0 = y_1 : 1 = -px_1 : -b.$$

These equations are satisfied by no values of x_1 and y_1. Hence *a diametral line has no pole.*

Combining these results with those already stated for the ellipse and hyperbola, we have the theorem:

With respect to any given conic a diametral line (*or an asymptote*) *has no pole, but every other line has a unique pole.*

EXERCISES

With respect to each of the following conics find the pole of the given line and draw the figure.

1. $x^2 + y^2 = 36$,	$3x + y = 6$.
2. $x^2 + y^2 = 64$,	$x - y = 15$.
3. $9x^2 + 25y^2 = 225$,	$x + 2y = 4$.
4. $25x^2 + 9y^2 = 225$,	$x - 2y = 15$.
5. $x^2 - y^2 = 100$,	$x + 2y = 3$.
6. $9x^2 - 25y^2 = 225$,	$x + y = 0$.
7. $y^2 = 16x$,	$x - y = 2$.
8. $y^2 = -8x$,	$4x - y = 4$.

9. $(x-4)^2+(y-3)^2=25, \quad x-y=0.$

10. $(y-2)^2=8(x-3), \quad x=0;$ also $y=0.$

11. $x^2+4y^2+8x=0, \quad x=-8;$ also $y=1.$

12. $xy=16, \quad x+y=8.$

13. Prove that with respect to a parabola the pole of a line which passes through the focus is a point on the directrix. Is there any exception?

14. Prove that with respect to an ellipse the pole of a line which passes through a focus is a point on the corresponding directrix. Is there any exception?

15. With respect to the ellipse

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$$

find the pole P_1 of the line $y=k$. Discuss the motion of P_1 as k approaches zero through positive values; also through negative values. Do the same for the line $x=k$.

16. With respect to the parabola

$$y^2=2px$$

find the pole P_1 of the line $y=mx$. Discuss the motion of P_1 as m approaches zero. Do the same for the line $y=mx+2$.

103. Harmonic division. If two points P and Q divide a line segment MN externally and internally in ratios having equal numerical values, so that

$$(1) \qquad \frac{MP}{PN}=-\frac{MQ}{QN},$$

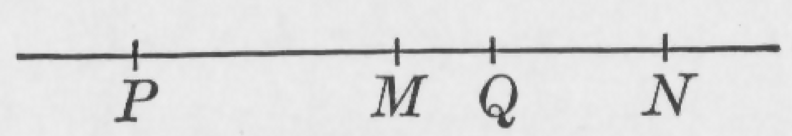


FIG. 110

then P and Q are said to divide MN **harmonically,** and P and Q are called **harmonic conjugates** with respect to MN.

From (1) it follows that

$$(2) \qquad \frac{PM}{MQ}=-\frac{PN}{NQ}.$$

Hence *if* P *and* Q *divide* MN *harmonically, then* M *and* N *divide* PQ *harmonically.*

It can also be shown that PQ is what is called in algebra a *harmonic mean* between PM and PN; that is, that

$$\frac{1}{PM}+\frac{1}{PN}=\frac{2}{PQ}.$$

If we write

$$\frac{MP}{PN}=\frac{r_1}{r_2},$$

we see from equation (1) that when P divides MN in the ratio $r_1:r_2$, Q divides MN in the ratio $-r_1:r_2$. If the coordinates of M, N, P, Q are indicated as follows,

$$(3) \qquad M(x_1,y_1), \quad N(x_2,y_2), \quad P(x_3,y_3), \quad Q(x_4,y_4),$$

we have, by the formulas for a point of division, page 44,

$$(4) \qquad \begin{aligned} x_3&=\frac{r_1x_2+r_2x_1}{r_1+r_2}, & y_3&=\frac{r_1y_2+r_2y_1}{r_1+r_2},\\ x_4&=\frac{-r_1x_2+r_2x_1}{-r_1+r_2}, & y_4&=\frac{-r_1y_2+r_2y_1}{-r_1+r_2}.\end{aligned}$$

We now state an important property of a pole P and its polar L with respect to a given conic C:

If a line through P *intersects the conic* C *at points* M *and* N *and the polar* L *at* Q, *then* M *and* N *divide* PQ *harmonically.*

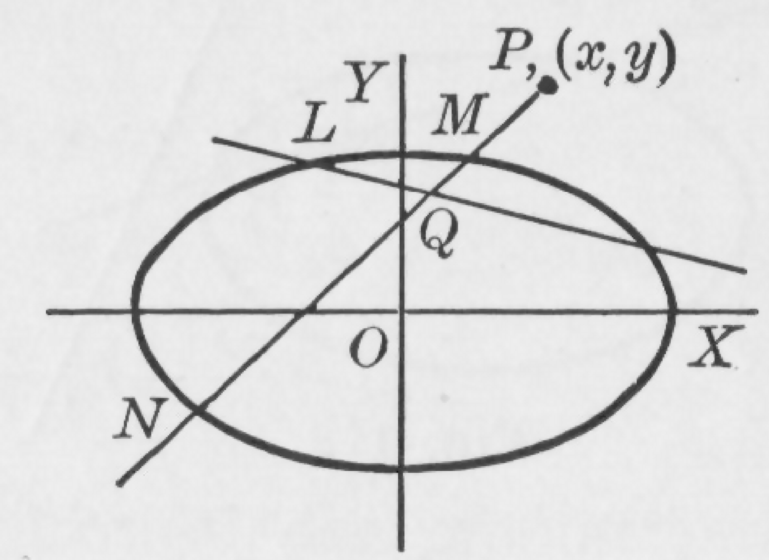


FIG. 111

We shall prove this theorem for the case where the conic C is an ellipse or hyperbola. When axes are suitably chosen the equation of C is

$$(5) \qquad Ax^2+By^2=1,$$

and the polar L of $P(x_3,y_3)$ is

$$(6) \qquad Axx_3+Byy_3=1.$$

Let the coördinates of M and N be (x_1, y_1) and (x_2, y_2) respectively, and choose r_1 and r_2 such that $MP/PN = r_1/r_2$. Then the first two equations of (4) are verified, and it will suffice to prove that if the last two are also verified the point (x_4, y_4) lies on the polar of P. This is the same as saying that (x_4, y_4) can be substituted for (x, y) in (6); that is, we are to prove that

$$(7) \qquad Ax_4x_3 + By_4y_3 = 1.$$

By using equations (4) we at once reduce (7) as follows:

$$A\,\frac{r_2{}^2x_1{}^2 - r_1{}^2x_2{}^2}{r_2{}^2 - r_1{}^2} + B\,\frac{r_2{}^2y_1{}^2 - r_1{}^2y_2{}^2}{r_2{}^2 - r_1{}^2} = 1,$$

$$(8) \qquad \frac{r_2{}^2}{r_2{}^2 - r_1{}^2}\,(Ax_1{}^2 + By_1{}^2) - \frac{r_1{}^2}{r_2{}^2 - r_1{}^2}\,(Ax_2{}^2 + By_2{}^2) = 1.$$

But since (x_1, y_1) and (x_2, y_2) are on the conic, each satisfies (5), and each expression in parentheses in (8) is equal to 1. We at once verify equation (8), and our proof is thus completed.

104. Relations of poles and polars. Theorems concerning poles and polars may be stated in pairs as illustrated in the following propositions.*

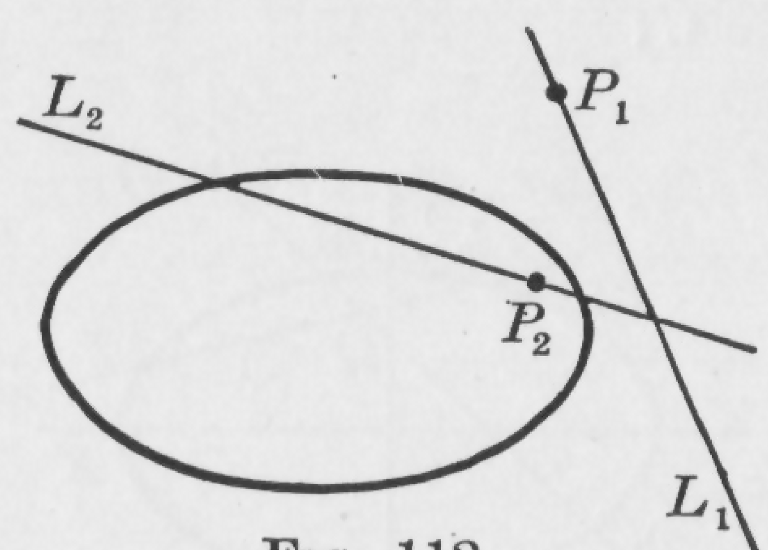


FIG. 112

Theorem 1a. If, for a given conic, two points P_1 *and* P_2 *are so located that* P_2 *lies on the polar of* P_1*, then* P_1 *lies on the polar of* P_2.

Theorem 1b. If, for a given conic, two lines L_1 *and* L_2 *are so located that* L_2 *passes through the pole of* L_1*, then* L_1 *passes through the pole of* L_2.

* In the following theorems only those points and lines which have polars and poles are considered.

Let us prove Theorem 1a for the case in which the conic is an ellipse or hyperbola. When axes are properly chosen the equation of the conic has the form

$$(1) \qquad Ax^2 + By^2 = 1.$$

The polars of $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$ are respectively

$$(2) \qquad Ax_1x + By_1y = 1,$$

$$(3) \qquad Ax_2x + By_2y = 1.$$

By hypothesis P_2 lies on the line (2); hence

$$Ax_1x_2 + By_1y_2 = 1.$$

We are to prove that P_1 lies on the line (3); that is, that

$$Ax_2x_1 + By_2y_1 = 1.$$

Since the last two equations are identical, the proposition follows.

To prove Theorem 1b for the case of the conic (1), let L_1 and L_2 be the lines

$$(4) \qquad a_1x + b_1y = 1,$$

$$(5) \qquad a_2x + b_2y = 1,$$

whose poles are, by § 102, $P_1(a_1/A, b_1/B)$ and $P_2(a_2/A, b_2/B)$. By hypothesis, line (5) goes through P_1; hence

$$a_2a_1/A + b_2b_1/B = 1.$$

We are to prove that line (4) passes through P_2; that is, that

$$a_1a_2/A + b_1b_2/B = 1.$$

Since the last two equations are identical, the proposition follows.

A second pair of theorems is the following:

Theorem 2a. For a given conic, the pole of a line joining two points P_1 *and* P_2 *is the point* P *of intersection of the polars of* P_1 *and* P_2.

Theorem 2b. For a given conic the polar of the point of intersection of two lines L_1 *and* L_2 *is the line* L *joining the poles of* L_1 *and* L_2.

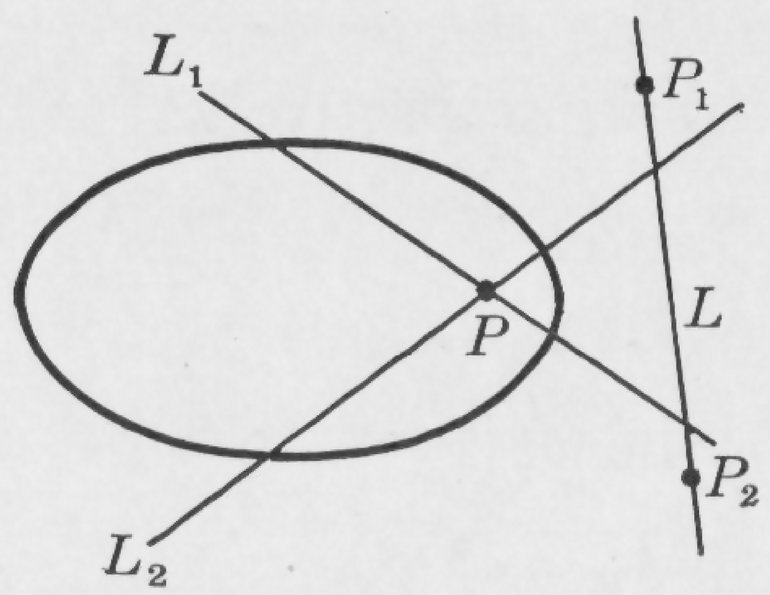


FIG. 113

To prove the former of these theorems, we use Theorem 1*a*. The pole of P_1P_2 lies on the polar of P_1; it also lies on the polar of P_2. Hence it is the point of intersection of these polars.

The other theorem is proved similarly.

Another pair of theorems is as follows:

Theorem 3a. If a number of lines all pass through a point P, *their poles with respect to a given conic all lie on a line.*

Theorem 3b. If a number of points lie on a line L, *their polars with respect to a given conic all pass through one point.*

It is seen at once that, in Theorem 3*a*, the poles all lie on the polar of P, and in Theorem 3*b* the polars all pass through the pole of L.

105. The principle of duality. As illustrated in § 104, theorems concerning poles and polars occur in pairs. We may obtain one theorem from another by the following interchanges of words and phrases:

point with *line*; *pole* with *polar*;
lies on with *passes through*;
point of intersection of with *line joining*.

This is known as the **principle of duality.** A theorem obtained from a given theorem by such an interchange is called the **dual** of the given theorem.

EXERCISES

1. Prove that if points P and Q divide MN harmonically, then PQ is a harmonic mean between PM and PN (see page 223).

2. Prove that if a line through P intersects a parabola at points M and N and cuts the polar of P at Q, then M and N divide PQ harmonically.

3. Prove Theorem 1*a*, § 104, in case the conic is a parabola.

4. Prove Theorem 1*b*, § 104, in case the conic is a parabola.

5. Prove that for a circle of radius a and center O, if the polar of P_1 intersects OP_1 at P_2, then $OP_1 \cdot OP_2 = a^2$.

6. Prove that the polar of a point with respect to a circle is perpendicular to the diameter (produced if necessary) which passes through the point.

7. Prove that, for a circle of radius a and center O, the polar of an exterior point P_1 is the common chord of the given circle and the circle having OP_1 as a diameter.

8. For the conic $y^2 = 8x$, find the pole P_1 of the line which passes through $P_2(2, 2)$ and $P_3(4, 4)$. Also find the equations of the polars of P_2 and P_3 and verify that they pass through P_1.

9. For the curve
$$x^2 + 4y^2 = 36,$$
find the coördinates of the pole P_1 of the line which passes through the points $P_2(2, 0)$ and $P_3(0, 4)$. Find also the equations of the polars of P_2 and P_3 and verify that P_1 lies on both of them.

10. For the curve
$$x^2 - 4y^2 = 36,$$
find the equations of the polars of the three points $P_1(2, 4)$, $P_2(4, 6)$, $P_3(6, 8)$ and find the coördinates of their point of intersection.

11. Prove that, for a parabola, the pole of the normal at an end of the latus rectum lies on a diametral line which passes through the other end of the latus rectum.

12. Prove that the polar of a point P_1 with respect to a hyperbola whose center is O is perpendicular to the line OP_1 if and only if P_1 is on the transverse or conjugate axis of the hyperbola.

13. Find the conditions under which the polar of a point P_1 with respect to an ellipse whose center is O is perpendicular to the line OP_1.

14. Let L_1 be the polar of P_1 with respect to a parabola, and let the diametral line through P_1 cut L_1 at P_2 and cut the parabola at Q. Prove that $P_1Q = QP_2$, and that L_1 is parallel to the tangent at Q.

15. Let L_1 be the polar of a point P_1 with respect to an ellipse whose center is O. Draw a diameter D through P_1, and its conjugate diameter D'. Prove that D' is parallel to L_1. Also prove that if the length of D is $2d$, and if the line OP_1 intersects L_1 in P_2, then

$$OP_1 \cdot OP_2 = d^2.$$

16. Let C and C' be a hyperbola and its conjugate. For a point P let L and L' be the polars with respect to C and C'. Let D be a diametral line of C through P, and let D' be its conjugate diametral line. Prove that L and L' are parallel to D'. Show that the distance from L to D' equals the distance from L' to D'.

CHAPTER XII

THE GENERAL EQUATION OF THE SECOND DEGREE

106. The problem of reduction to standard forms. The general equation of second degree may be written *

$$(1) \qquad ax^2 + 2bxy + cy^2 + 2dx + 2ey + f = 0,$$

where the coefficients a, b, c are not all zero. In various places in this book we have already studied particular forms of equation (1); in Chapters V and VI certain types have been designated *standard forms* for circles, parabolas, ellipses, and hyperbolas.

In Chapter VII we have used transformations of coördinates corresponding to translation and to rotation of the coördinate axes in order to simplify certain equations of type (1). In §§ 107–110 of the present chapter we shall use such transformations in order to reduce *all* equations (1) to equations in the new coördinates which are either the standard forms for conic sections or closely related forms. By extending our definition of the term *conic section* to include cases where (1) is factorable, and where its locus consists of but one point, we shall be able to prove the following theorem:

Every equation of the second degree which has a locus represents a conic section.

In §§ 111, 112 we shall investigate certain expressions in terms of the coefficients in (1), called *invariants*, whose

* The coefficient of the term in xy is written $2b$ instead of b in order to simplify certain formulas in the following sections of this chapter. For similar reasons we write $2d$ and $2e$ instead of d and e.

values enable us to determine the type of locus represented by any given equation (1).

107. Removal of terms by translation of axes. On page 143 it was shown that the equations for translating the axes to the new origin (h, k) are

$$(1) \qquad x = x' + h, \qquad y = y' + k.$$

With the substitutions thus indicated, equation (1) of the preceding section becomes

$$a(x' + h)^2 + 2b(x' + h)(y' + k) + c(y' + k)^2 + 2d(x' + h) + 2e(y' + k) + f = 0.$$

When this equation has been simplified it may be written

$$(2) \quad ax'^2 + 2bx'y' + cy'^2 + 2d'x' + 2e'y' + f' = 0,$$

where

$$(3) \qquad \begin{aligned} d' &= ah + bk + d, \\ e' &= bh + ck + e, \\ f' &= ah^2 + 2bhk + ck^2 + 2dh + 2ek + f. \end{aligned}$$

If we can choose h and k so that d' and e' are both zero, equation (2) will have an especially simple form. The equations $d' = 0$, $e' = 0$ are equivalent to the pair

$$(4) \qquad \begin{aligned} ah + bk + d &= 0, \\ bh + ck + e &= 0. \end{aligned}$$

We can solve equations (4) by the formulas

$$(5) \qquad h = \frac{cd - be}{b^2 - ac}, \qquad k = \frac{ae - bd}{b^2 - ac},$$

provided $b^2 - ac \neq 0$.

The following calculation gives us the value of f' when h and k have the values (5). The last of equations (3) can be written

$$f' = h(ah + bk + d) + k(bh + ck + e) + dh + ek + f.$$

By equations (4), f' reduces to

$$(6) \qquad f' = dh + ek + f.$$

When the values (5) for h and k are substituted in (6) we obtain *

$$(7) \qquad f' = -\frac{\Delta}{b^2 - ac},$$

where †

$$\Delta = [d(be - cd) - e(ae - bd) + f(ac - b^2)]$$

$$(8) \qquad = \begin{vmatrix} a & b & d \\ b & c & e \\ d & e & f \end{vmatrix}.$$

We summarize these results as follows:

If $b^2 - ac \neq 0$, *the equation*

$$ax^2 + 2bxy + cy^2 + 2dx + 2ey + f = 0,$$

can be reduced to the form

$$(9) \qquad ax'^2 + 2bx'y' + cy'^2 + f' = 0$$

by a translation of axes

$$x = x' + h, \quad y = y' + k,$$

where h *and* k *(given by formulas (5)) are solutions of equations (4), and where* f′ *is given by formulas (7) and (8), or can be computed from (6), and (5) or (4).*

It is to be noted that if b is zero, (9) can be at once put into standard forms. Further reduction of (9) by rotation of axes in case $b \neq 0$ will be considered in § 109.

Example. — Reduce the equation

$$x^2 - 4y^2 + 6x + 8y - 11 = 0$$

to a standard form by translating the axes.

* The symbol in the numerator of formula (7) is the capital Greek letter "delta."

† To see that the determinant form given for Δ reduces to the preceding expression in brackets, note that this expression is the expansion of the determinant in terms of the minors of the elements of its last column (see page 2).

Solution. — For the given equation

$$x^2 - 4y^2 + 6x + 8y - 11 = 0,$$

equations (4) become

$$h + 3 = 0, \quad - 4k + 4 = 0,$$

giving $h = -3$, $k = 1$; and from (6)

$$f' = 3h + 4k - 11 = -16.$$

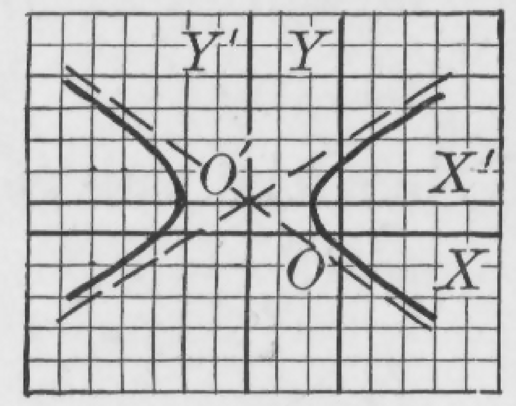


Fig. 114

It follows that the transformation

$$x = x' - 3, \quad y = y' + 1 \tag{10}$$

reduces the given equation to the equation of form (9)

$$x'^2 - 4y'^2 - 16 = 0.$$

In standard form, this last equation is

$$\frac{x'^2}{4^2} - \frac{y'^2}{2^2} = 1. \tag{11}$$

The curve is the hyperbola shown in Figure 114, where the old and the new axes, as well as the asymptotes, are also shown. Coördinates of such points as the center, foci, vertices, and equations of axes, directrices, asymptotes, can be written down in terms of x' and y', and then expressed in terms of x and y by means of equations (10). For example, the foci are

$$x' = \pm 2\sqrt{5}, \qquad y' = 0.$$

If these values are substituted in equations (10), we find that the x, y coördinates of the foci are

$$x = \pm 2\sqrt{5} - 3, \quad y = 1.$$

Similarly the asymptotes are

$$\frac{x'^2}{4^2} - \frac{y'^2}{2^2} = 0,$$

or

$$\frac{(x+3)^2}{4^2} - \frac{(y-1)^2}{2^2} = 0.$$

EXERCISES

Find the point (h, k) *to which the origin must be moved in a translation of axes in order to remove the terms of first degree from each of the following equations. Obtain the new equation.*

1. $x^2 + y^2 + 4x - 2y = 3.$

2. $2x^2 + 2y^2 - y = 12.$

3. $x^2 - 4y^2 - 2x + 8y + 3 = 0.$

4. $3xy - 2x + 4y + 1 = 0.$

5. $3x^2 + 4xy + 4x = 2.$

6. $2xy - y^2 + 8y = 1.$

7. $8x^2 - 6xy + y^2 + 2x + 2y + 1 = 0.$

8. $x^2 - 2xy + 5y^2 + 6x - 2y + 3 = 0.$

9. $2x^2 - 5xy + 2y^2 - 17x + 19y = 0.$

10. $x^2 + xy + y^2 + x - y = 1.$

For each of the following equations, translate the axes so as to remove the terms of first degree, and reduce the new equation to a standard form. Draw a figure showing the curve and both the old and the new axes. Find coördinates of points and equations of lines in the original x, y *system as indicated for each problem.*

11. $x^2 + 4y^2 - 2x + 4y - 2 = 0.$ Find coördinates of center and of foci.

12. $xy - x - y = 0.$ Find equations of asymptotes.

13. $9x^2 - y^2 + 2y = 10.$ Find equations of the line on which the transverse axis lies, of directrices, and of asymptotes.

14. $x^2 - 4y^2 + 2x + 8y + 9 = 0.$ Find coördinates of center and equations of asymptotes.

15. $9x^2 + 4y^2 - 36x + 8y + 4 = 0.$ Find equations of the lines on which the axes lie and of directrices.

16. $4x^2 + 16y^2 - 4x + 16y = 11.$ Find coördinates of center, of foci, and of vertices.

108. Removal of the *xy* term by rotation of axes. We now consider how the equation

$$ax^2 + 2bxy + cy^2 + 2dx + 2ey + f = 0 \tag{1}$$

may be simplified by a transformation

$$\begin{aligned} x &= x' \cos\theta - y' \sin\theta, \\ y &= x' \sin\theta + y' \cos\theta, \end{aligned} \tag{2}$$

which in Chapter VII, page 147, has been interpreted as a rotation of axes through the angle θ.

If the substitutions (2) are made in equation (1) and the result is simplified, we obtain the transformed equation

$$(3) \qquad A'x'^2 + 2B'x'y' + C'y'^2 + 2D'x' + 2E'y' + f = 0,$$

where

$$(4) \qquad \begin{aligned} A' &= a\cos^2\theta + 2b\sin\theta\cos\theta + c\sin^2\theta, \\ B' &= -a\sin\theta\cos\theta + b(\cos^2\theta - \sin^2\theta) + c\sin\theta\cos\theta, \\ C' &= a\sin^2\theta - 2b\sin\theta\cos\theta + c\cos^2\theta, \\ D' &= d\cos\theta + e\sin\theta, \\ E' &= -d\sin\theta + e\cos\theta. \end{aligned}$$

We now choose θ so that we shall have $B' = 0$. From the second of equations (4), the angle θ is then a solution of

$$(5) \qquad -a\sin\theta\cos\theta + b(\cos^2\theta - \sin^2\theta) + c\sin\theta\cos\theta = 0.$$

If we divide this equation through by $-\cos^2\theta$, and, for brevity, write t for $\tan\theta$, we obtain the equation

$$(6) \qquad bt^2 + (a - c)t - b = 0 \quad (t = \tan\theta).$$

When $b = 0$, we take $t = 0$. If $b \neq 0$, equation (6) has the two real solutions

$$t = \frac{-(a - c) \pm \sqrt{(a - c)^2 + 4b^2}}{2b}.$$

Since $t = \tan\theta = \sin\theta/\cos\theta$, we have

$$(7) \qquad \sin\theta = t\cos\theta.$$

We also have

$$(8) \qquad \sin^2\theta + \cos^2\theta = 1.$$

Equations (7) and (8) are satisfied if we take*

$$(9) \qquad \sin\theta = \frac{t}{\sqrt{1 + t^2}}, \qquad \cos\theta = \frac{1}{\sqrt{1 + t^2}}.$$

We summarize these results in the following theorem:

* It is possible to make other choices of $\sin\theta$ and $\cos\theta$, in which signs are different from those in (9), but we are interested only in finding one transformation which is effective.

The equation

$$(1) \qquad ax^2 + 2bxy + cy^2 + 2dx + 2ey + f = 0$$

can be reduced to the form

$$A'x'^2 + C'y'^2 + 2D'x + 2E'y + f = 0$$

by a rotation of axes

$$(2) \qquad \begin{aligned} x &= x'\cos\theta - y'\sin\theta, \\ y &= x'\sin\theta + y'\cos\theta. \end{aligned}$$

For this purpose we may take $\sin\theta$, $\cos\theta$ *as follows:*

$$(9) \qquad \sin\theta = \frac{t}{\sqrt{1 + t^2}}, \qquad \cos\theta = \frac{1}{\sqrt{1 + t^2}},$$

where t *is a root of the equation*

$$(6) \qquad bt^2 + (a - c)t - b = 0.$$

Example 1. — By a rotation of axes, transform the equation

$$5x^2 + 6xy + 5y^2 = 8$$

so as to remove the xy term.

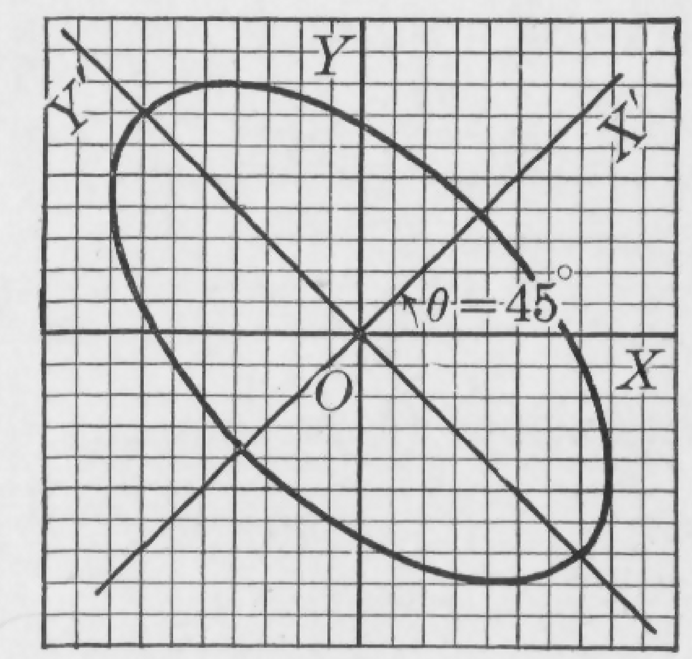


FIG. 115

Solution. — Here we have

$$a = 5, \quad b = 3, \quad c = 5, \quad d = 0, \quad e = 0, \quad f = -8.$$

A solution of equation (6) is $t = \tan\theta = 1$; hence we can take θ equal to $45°$. If we substitute the values $\sin\theta = 1/\sqrt{2}$, $\cos\theta = 1/\sqrt{2}$ in equations (4), we have

$$A' = 8, \quad B' = 0, \quad C' = 2, \quad D' = 0, \quad E' = 0,$$

and the transformed equation is $8x'^2 + 2y'^2 - 8 = 0$, or

$$(10) \qquad \frac{x'^2}{1} + \frac{y'^2}{4} = 1.$$

The curve is an ellipse whose semi-axes are of lengths 1 and 2. The curve and the two sets of axes are shown in Figure 115. The transformation (2) is

$$(11) \qquad x = \frac{x' - y'}{\sqrt{2}}, \qquad y = \frac{x' + y'}{\sqrt{2}}.$$

We can find the coördinates of the center, foci, and vertices, or the equations of lines of axes and of directrices in the x', y' system by inspection of (10). Equations (11) can be used to express these coördinates and equations in the x, y system. For example, the directrices are $y' = \pm 4/\sqrt{3}$. If we solve equations (11) for y' we have $y' = (y - x)/\sqrt{2}$. Hence in the old coördinates the directrices are

$$y - x = \pm 4\sqrt{\frac{2}{3}}.$$

Example 2. — By a rotation of axes transform the equation

$$9x^2 - 24xy + 16y^2 - 20x = 50$$

so that the $x'y'$ term is absent from the new equation.

Solution. — We have here

$$a = 9, \quad b = -12, \quad c = 16, \quad d = -10, \quad e = 0, \quad f = -50.$$

Equation (6) is

$$-12t^2 - 7t + 12 = 0,$$

which has the solution $t = 3/4$. With this value of t we have, from equations (9),

$$\sin\theta = \frac{3}{5}, \quad \cos\theta = \frac{4}{5}.$$

Equations (4) become

$$A' = 0, \quad B' = 0, \quad C' = 25, \quad D' = -8, \quad E' = 6.$$

The new equation is

$$25y'^2 - 16x' + 12y' = 50.$$

The equations of transformation are

$$x = \frac{1}{5}(4x' - 3y'), \qquad y = \frac{1}{5}(3x' + 4y').$$

EXERCISES

For each of the following equations obtain the equations of a rotation of axes that will remove the xy *term. Write the new equation, giving proper numerical values for the coefficients.*

1. $x^2 + 4xy + y^2 + 2x = 4$.

2. $x^2 - xy + y^2 + 4y = 0$.

3. $4x^2 - 24xy - 3y^2 + 5x = 0$.

4. $16x^2 + 24xy + 9y^2 + x + 2y = 1$.

5. $x^2 + 2xy + y^2 - 8x + 8y = 4$.

6. $2xy + 2x + 4y + 3 = 0$.

7. $4x^2 + 24xy + 11y^2 + 10x - 15y = 1$.

8. $3x^2 - 4xy - \sqrt{5}x = 4$.

9. $9x^2 - 24xy + 41y^2 = 0$.

10. $4x^2 - 2xy + 2y^2 - x = 0$.

For each of the following equations, rotate the axes so as to remove the xy *term, and reduce the new equation to a standard form. Draw a figure showing the curve and both the old and the new axes. Find coördinates of points and equations of lines in the original* x, y *system as indicated for each problem.*

11. $5x^2 + 6xy + 5y^2 = 8$. Find coördinates of center and of foci.

12. $3x^2 - 10xy + 3y^2 = 32$. Find equations of asymptotes.

13. $7x^2 + 48xy - 7y^2 + 25 = 0$. Find equations of directrices.

14. $(12x - 5y)^2 - 52(5x + 12y) = 0$. Find coördinates of vertex and of focus.

15. $6x^2 + 12xy + y^2 = 30$. Find coördinates of center and of vertices.

16. $17x^2 + 30xy + 17y^2 = 32$. Find equations of lines of axes and of directrices.

109. Reduction of equations of central type. According to the theorem of page 231, the general equation of second degree

$$(1) \qquad ax^2 + 2bxy + cy^2 + 2dx + 2ey + f = 0$$

can be reduced, by an appropriate translation of axes, to the form

$$(2) \qquad ax'^2 + 2bx'y' + cy'^2 + f' = 0,$$

provided $b^2 - ac \neq 0$. The locus of equation (2) is symmetrical to the new origin $x' = 0$, $y' = 0$, since the point $(-x', -y')$ is on the curve whenever (x', y') is so situated. The curve is therefore said to have the **center** $x' = 0$, $y' = 0$, and the equation is said to be of **central type.** Note that if $b^2 - ac$ is not zero, the expression composed of the terms of second degree in (1) and (2) cannot be a perfect square.

To reduce an equation (1) of central type to a standard form we first reduce it to an equation (2) by a translation of axes as explained in § 107. We then rotate the x', y' axes into a new x'', y'' set by the methods of § 108, so that the final equation lacks the $x'y'$ term, and hence is of form

$$(3) \qquad Ax''^2 + Cy''^2 + f' = 0,$$

where f' is given by equations (7) and (8) of § 107 (page 231). Equation (3) is easily reduced to standard forms.

We now show that *neither* A *nor* C *can be zero.* Equation (2) can be obtained from (3) by rotating the axes backward. If A and C were both zero, (2) would have no terms of second degree. If either A or C were zero, the terms of second degree in (3) would form a perfect square, and the same would be true of the terms of second degree of (2) and (1); this is impossible, since (1) is of central type.

The coefficient f' can, however, be zero. If $f' \neq 0$, an equivalent form for (3) is

$$(4) \qquad \frac{x''^2}{-f'/A} + \frac{y''^2}{-f'/C} = 1.$$

Equation (4) represents an **ellipse, no locus,** or a **hyperbola,** according as the two quantities $-f'/A$, $-f'/C$ are both positive, both negative, or of opposite sign.

If $f' = 0$, equation (3) can be factored provided A and C are of opposite sign, in which case the locus is a **pair of intersecting lines.** If A and C are of the same sign, the locus of (3) is a **point-ellipse** whose only real point is $x'' = 0$, $y'' = 0$.

The loci for which $f' = 0$ are sometimes called **degenerate conics.** Note that these conics are also sections of a right circular cone, the cutting plane here passing through the vertex.

We have thus established the theorem stated in § 106 (page 229), for equations (1) of central type; all such equations which have a locus represent conic sections.

Example. — Reduce the equation

$$(5) \qquad 8x^2 + 12xy + 17y^2 - 28x - 46y + 17 = 0$$

to a standard form, and plot the curve.

Solution. — We first translate the origin so as to remove the terms of first degree. Equations (4) of § 107 (page 230) are, for this Example

$$8h + 6k - 14 = 0,$$
$$6h + 17k - 23 = 0,$$

giving $h = 1$, $k = 1$. By equation (6) of § 107 (page 231), we have $f' = -20$. Hence the translation of axes

$$(6) \qquad x = x' + 1, \qquad y = y' + 1$$

transforms (5) into

$$(7) \qquad 8x'^2 + 12x'y' + 17y'^2 - 20 = 0.$$

To remove the $x'y'$ term from (7) we rotate the axes through the angle θ, where $\tan\theta = t$ is the positive solution of equation (6) of § 108. This equation for the present Example is

$$6t^2 - 9t - 6 = 0,$$

giving $t = 2$. Hence the transformation (2) of the theorem of § 108 (page 235) is, with a slight change of notation,

$$(8) \qquad x' = \frac{x'' - 2y''}{\sqrt{5}}, \qquad y' = \frac{2x'' + y''}{\sqrt{5}}.$$

When we make the corresponding substitutions in (7), the latter equation reduces to

$$20x''^2 + 5y''^2 - 20 = 0,$$

or

$$(9) \qquad \frac{x''^2}{1} + \frac{y''^2}{4} = 1.$$

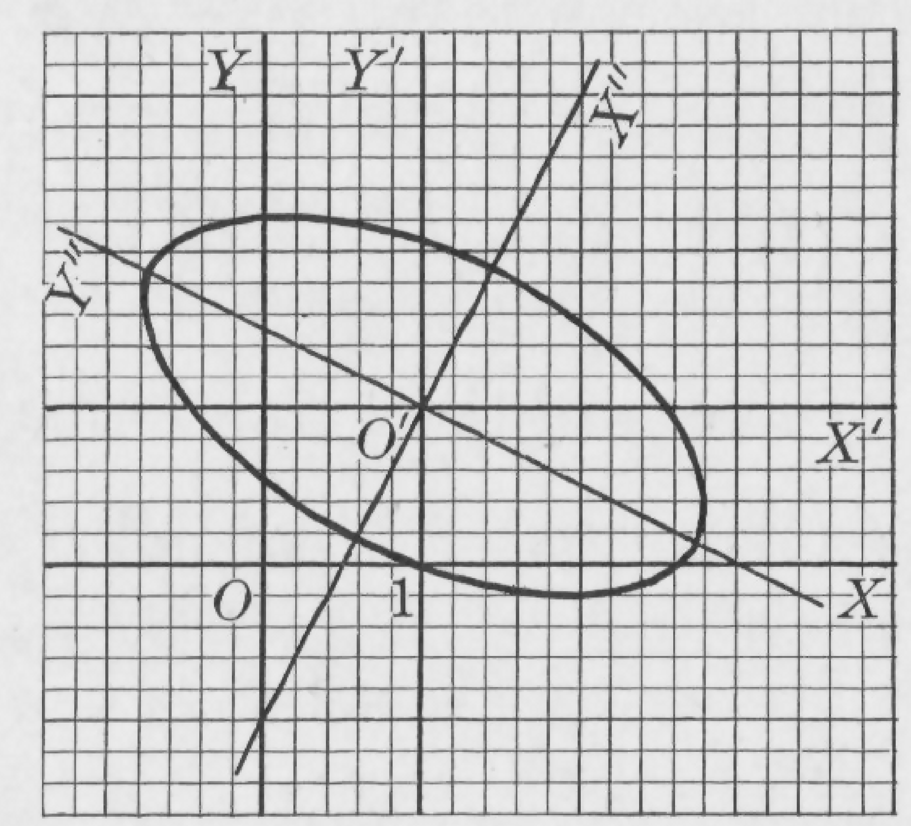


Fig. 116

This ellipse is readily drawn in the x'', y'' system. Figure 116 shows the curve and the three sets of axes we have used. The axes $O'X'$, $O'Y'$ pass through the point $x = h = 1$, $y = k = 1$, and the angle from $O'X'$ to $O'X''$ is θ, where $\tan\theta = 2$; thus $O'X''$ is constructed as a line through O' with slope 2 in the x', y' system, and we do not need to look up θ in the tables.

Coördinates of center, foci, and vertices, or equations of axes and directrices, can be obtained for the x'', y'' system by the methods of

Chapter VI. We can reduce these coördinates and equations to expressions in x and y by means of equations (8) and (6).

110. Reduction of equations of parabolic type. The general equation of the second degree

$$(1) \qquad ax^2 + 2bxy + cy^2 + 2dx + 2ey + f = 0,$$

is said to be of **parabolic type** if $b^2 - ac = 0$. We shall now show how in this case (1) is reduced to a standard form for a parabola, or to related forms.

Since $b^2 = ac$, the coefficients a and c are not of opposite sign; if both were negative we could make them positive by multiplying equation (1) by -1, hence we shall take a and c positive (or one of them may be zero). Let us write

$$(2) \qquad a = \alpha^2, \quad 2b = 2\alpha\beta, \quad c = \beta^2.$$

Equation (1) then becomes

$$(3) \qquad (\alpha x + \beta y)^2 + 2dx + 2ey + f = 0.$$

Let us first rotate the axes so as to remove the xy term in case $b \neq 0$. By the theorem of § 108 (page 235), we find that we can take $t = -\alpha/\beta$, and the equations (2) of the theorem can be written

$$(4) \qquad x = \frac{\beta x' + \alpha y'}{\sqrt{\alpha^2 + \beta^2}}, \qquad y = \frac{-\alpha x' + \beta y'}{\sqrt{\alpha^2 + \beta^2}}.$$

By means of these transformations (3) becomes

$$(5) \qquad (\alpha^2 + \beta^2)y'^2 + 2D'x' + 2E'y' + f = 0,$$

where

$$(6) \qquad D' = \frac{d\beta - e\alpha}{\sqrt{\alpha^2 + \beta^2}}, \qquad E' = \frac{d\alpha + e\beta}{\sqrt{\alpha^2 + \beta^2}}.$$

Even when $b = 0$, the transformation (4) can be used, and (5) and (6) are valid.

We distinguish two cases, the first where $D' \neq 0$, the second where $D' = 0$.

If $D' \neq 0$, the translation of axes

$$(7) \qquad \begin{aligned} x' &= x'' - \frac{1}{2D'}\left(f - \frac{E'^2}{\alpha^2 + \beta^2}\right), \\ y' &= y'' - \frac{E'}{\alpha^2 + \beta^2}, \end{aligned}$$

changes (5) into

$$(8) \qquad (\alpha^2 + \beta^2)y''^2 + 2D'x'' = 0.$$

A computation of the expression Δ (formula (8) of § 107, page 231) for equation (3) gives

$$\Delta = -(d\beta - e\alpha)^2,$$

so that we have, since $\alpha^2 = a$, $\beta^2 = c$,

$$(9) \qquad D' = \pm\sqrt{\frac{-\Delta}{a + c}},$$

and we write (8) in the form

$$(10) \qquad (a + c)y''^2 \pm 2\sqrt{\frac{-\Delta}{a + c}}\,x'' = 0.$$

This can at once be reduced to a standard form for the parabola. Note that the condition $D' \neq 0$ is equivalent to $\Delta \neq 0$.

If $D' = 0$, that is (from (9)) if $\Delta = 0$, the translation

$$x' = x'', \qquad y' = y'' - \frac{E'}{\alpha^2 + \beta^2}$$

changes (5) to the form

$$(11) \qquad (a + c)y''^2 + F' = 0,$$

where F' can readily be expressed in terms of the coefficients of (1).* Equation (11) has no locus if $F' > 0$, since

* We have

$$F' = f - \frac{E'^2}{a + c} = f - \frac{1}{a + c}\left(E'^2 - \frac{\Delta}{a + c}\right) \quad (\text{since } \Delta = 0)$$
$$= f - \frac{d^2 + e^2}{a + c}.$$

$a + c > 0$; it represents two parallel lines if $F' < 0$, or the single line $y''^2 = 0$ if $F' = 0$.

Two parallel lines are a section of a cylinder, which is a limiting case of a cone. If we regard them, as well as the line $y''^2 = 0$, as degenerate conics, we have completed the proof of the theorem of § 106, for we have shown that second degree equations of parabolic type that have a locus, as well as those of central type, are conic sections, the degenerate conics being included under that term.

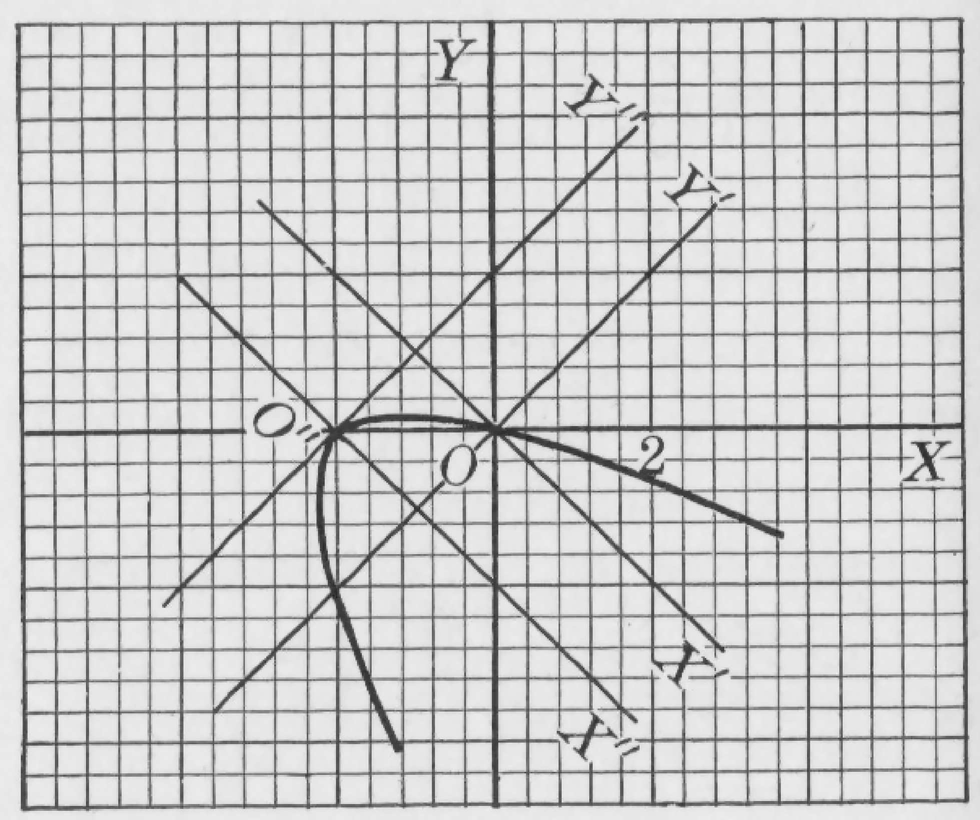


FIG. 117

Example.—Draw the graph of the equation

$x^2 + 2xy + y^2 + 2x + 6y = 0.$

Solution. — Here $\alpha = 1$, $\beta = 1$, $d = 1$, $e = 3$, $f = 0$. Equations (4) are

$$x = \frac{x' + y'}{\sqrt{2}}, \qquad y = \frac{-x' + y'}{\sqrt{2}}.$$

Equation (5) is

$$2y'^2 - \frac{4}{\sqrt{2}}x' + \frac{8}{\sqrt{2}}y' = 0.$$

Equations (7) are

$$x' = x'' - \sqrt{2}, \qquad y' = y'' - \sqrt{2},$$

and (8) is

$$2y''^2 - \frac{4}{\sqrt{2}}x'' = 0,$$

or

$$y''^2 = \sqrt{2}\,x''.$$

The curve and the three sets of axes are shown in Figure 117.

EXERCISES

Reduce each of the following equations to either form (3) *of* § 109, *page* 238, *or one of the forms* (8), (11) *of* § 110, *page* 241. *If the equation has a real locus draw the curve, indicating the three sets of coördinate axes.*

1. $5x^2 + 6xy + 5y^2 - 4x + 4y - 4 = 0.$
2. $13x^2 + 10xy + 13y^2 - 68x - 4y + 28 = 0.$

3. $3x^2 + 4xy + 6x + 4y = 1.$
4. $12xy - 5y^2 - 24x + 20y = 56.$
5. $x^2 - 2xy + y^2 - 4\sqrt{2}x - 4\sqrt{2}y - 8\sqrt{2} = 0.$
6. $9x^2 - 24xy + 16y^2 = 9\sqrt{5}(4x + 3y + 25).$
7. $3x^2 - 10xy + 3y^2 - 26x + 22y + 35 = 0.$
8. $11x^2 + 24xy + 4y^2 + 4x - 32y - 36 = 0.$
9. $3x^2 + 2xy + 3y^2 + 6x + 2y + 11 = 0.$
10. $5x^2 - 4xy + 2y^2 + 4x - 4y + 2 = 0.$
11. $7x^2 + 48xy - 7y^2 - 62x - 34y + 73 = 0.$
12. $3x^2 - 8xy - 3y^2 - 8x - 6y - 23 = 0.$
13. $4x^2 - 12xy + 9y^2 + 24x - 36y - 16 = 0.$
14. $4x^2 - 4xy + y^2 + 16x - 8y + 16 = 0.$
15. $17x^2 + 30xy + 17y^2 - 128x - 128y + 256 = 0.$
16. $144x^2 - 120xy + 25y^2 + 28x - 744y - 116 = 0.$

111. The invariants. Three expressions formed from the coefficients of the general conic

$$(1) \qquad ax^2 + 2bxy + cy^2 + 2dx + 2ey + f = 0$$

that have appeared in several places in the preceding sections are $a + c$, $b^2 - ac$, and Δ (see formula (8) of § 107, page 231). We shall now show that each of these is an **invariant** of (1) for change of axes; that is, if (1) is transformed * into

$$(2) \quad a''x''^2 + 2b''x''y'' + c''y''^2 + 2d''x'' + 2e''y'' + f'' = 0$$

by a change of axes, then we must have

$$a'' + c'' = a + c, \quad b''^2 - a''c'' = b^2 - ac, \quad \Delta'' = \Delta,$$

where Δ'' is the expression that corresponds to Δ for equation (2).

* In passing from (1) to (2) we understand that after making the substitutions at each step the resulting equation is not multiplied through by any number.

To prove that $a + c$ and $b^2 - ac$ are invariants we first note, from § 107, that after a translation the new coefficients a, b, c are the same as the old, hence $a + c$ and $b^2 - ac$ are unchanged. Let a rotation of axes change (1) into

$$(3) \quad A'x'^2 + 2B'x'y' + C'y'^2 + 2D'x' + 2E'y' + F' = 0.$$

Then the first three of equations (4), § 108 (page 234), give, after we have made use of the formulas $\sin^2\theta + \cos^2\theta = 1$, $\sin 2\theta = 2\sin\theta\cos\theta$, $\cos 2\theta = \cos^2\theta - \sin^2\theta$,

$$(4) \quad A' + C' = a + c,$$

$$(5) \quad A' - C' = (a - c)\cos 2\theta + 2b\sin 2\theta,$$

$$(6) \quad 2B' = -(a - c)\sin 2\theta + 2b\cos 2\theta.$$

If we square and add (5) and (6) we have

$$(7) \quad (A' - C')^2 + 4B'^2 = (a - c)^2 + 4b^2.$$

Square (4) and subtract from (7); we readily deduce the equation

$$(8) \quad B'^2 - A'C' = b^2 - ac.$$

Equations (4) and (8) show that $a + c$ and $b^2 - ac$ are invariant for a rotation of axes. Since these expressions are also unchanged by a translation, they are invariant for all changes of axes.

To show that Δ is invariant, we use an indirect method, since a direct calculation would be tedious. We first note that in §§ 109 and 110 we have shown that a suitable change of axes reduces the general equation (1) to one of the three forms

$$(9) \quad Ax''^2 + Cy''^2 - \frac{\Delta}{b^2 - ac} = 0 \quad \text{(formulas (3), § 109, and (7), § 107)},$$

$$(10) \quad (a + c)y''^2 \pm 2\sqrt{\frac{-\Delta}{a + c}}\,x'' = 0 \quad \text{(formula (10), § 110)},$$

$$(11) \quad (a + c)y''^2 + F' = 0 \quad \text{(formula (11), § 110, } \Delta = 0).$$

We now show that the expression Δ'' for each of the equations (9), (10), (11) is equal to Δ.

For (9) we have

$$a'' = A,\ b'' = 0,\ c'' = C,\ d'' = e'' = 0,\ f'' = -\frac{\Delta}{b^2 - ac}.$$

We find at once that

$$\Delta'' = AC\left(\frac{-\Delta}{b^2 - ac}\right).$$

However, on account of the invariancy of $b^2 - ac$, we have

$$(12) \quad b^2 - ac = b''^2 - a''c'' = -AC.$$

Hence $\Delta'' = \Delta$ for equation (9). For (10) we have

$$\Delta'' = -(a + c)\left[\sqrt{\frac{-\Delta}{a + c}}\right]^2 = \Delta,$$

and for (11), $\Delta'' = \Delta = 0$. Thus $\Delta'' = \Delta$ for each of the forms (9), (10), (11).

Now, since (2) is obtained from (1) by a change of axes, both (2) and (1) are transformable into the same equation (9) or (10) or (11). In the preceding paragraph we have proved that (1) has the same value of Δ as has the equation (9), (10), or (11) to which (1) is reducible, and the same is true of (2), hence Δ has the same value for (2) as for (1). This completes the proof of the invariancy of Δ.

We could similarly prove the invariancy of the expression F' in equation (11) when both $b^2 - ac$ and Δ are zero.

112. Applications of the invariants. In order to express all coefficients of the reduced forms (9), (10), (11) of § 111 in terms of the invariants we now proceed to examine the coefficients A and C in (9).

On account of the invariancy of $a + c$ we have

$$A + C = a + c,$$

and from (12) of § 111,

$$(1) \quad -AC = b^2 - ac.$$

From the identity

$$\begin{aligned}(r - A)(r - C) &\equiv r^2 - (A + C)r + AC \\ &\equiv r^2 - (a + c)r - (b^2 - ac)\end{aligned}$$

it follows that A and C are the solutions of the quadratic equation

$$(2) \qquad r^2 - (a + c)r - (b^2 - ac) = 0,$$

which may be written in determinant form

$$(2') \qquad \begin{vmatrix} a - r & b \\ b & c - r \end{vmatrix} = 0.$$

If the roots of (2) are r_1, r_2, we can take either root as A and the other as C; for if $r_1 = A$ and $r_2 = C$, then a further rotation of axes through 90°, for which the equations are $x'' = -y'''$, $y'' = x'''$, would change (9) of § 111 into an equation of the same form in which A and C are interchanged.

Since the equation of every conic can be reduced to one of the forms (9), (10), (11) of § 111 and since the coefficients of these reduced equations contain invariants only, it follows that every *intrinsic* property of a conic, that is, every property not dependent on choice of axes, is determined by the values of its invariants. Such intrinsic properties are, for example, its eccentricity, length of latus rectum, distance from focus to directrix, and lengths of axes if the conic is of central type. In particular, we can classify the types of locus representable by equation (1) of § 111 in the following table, which we shall first present, and then justify.

$\Delta \neq 0$ (non-degenerate types)	$\Delta = 0$ (degenerate types)
$b^2 - ac \neq 0$ (central types)	
1. $b^2 - ac < 0$ (elliptic types)	1. $b^2 - ac < 0$ (point ellipse)
(a) $(a + c)\Delta < 0$ (ellipse)	
(b) $(a + c)\Delta > 0$ (no locus)	
2. $b^2 - ac > 0$ (hyperbola)	2. $b^2 - ac > 0$ (intersecting lines)
$b^2 - ac = 0$ (parabolic types)	
1. (parabola)	1. $(a + c)F' < 0$ (parallel lines)
	2. $F' = 0$ (one line)
	3. $(a + c)F' > 0$ (no locus)

This table is verified by an examination of equations (9), (10), (11) of § 111. The central types are represented by (9), for which we recall that $AC = -(b^2 - ac)$. If we have $b^2 - ac < 0$, then A and C are of the same sign; and they are of opposite sign if $b^2 - ac > 0$. Note also that the condition $(a + c)\Delta < 0$ requires that $(a + c)$ and Δ be of opposite sign; similar remarks apply to the conditions $(a + c)\Delta > 0$, $(a + c)F' < 0$, $(a + c)F' > 0$. With these hints the student should be able to check the table for himself.

Note that for all the degenerate types, where $\Delta = 0$, the reduced equation, and therefore the original equation, has real or imaginary linear factors.

Example. — Determine the type of the curve which has the equation

$$3x^2 + 8xy - 3y^2 - 4x + 8y + 1 = 0,$$

and find its eccentricity.

Solution. — We have here

$$a + c = 0, \qquad b^2 - ac = 25, \qquad \Delta = -125.$$

The equation represents a hyperbola, since $\Delta \neq 0$, $b^2 - ac > 0$. Equation (2) is

$$r^2 - 25 = 0.$$

Hence the reduced form (9) of § 111 can be taken as

$$5x''^2 - 5y''^2 + 5 = 0.$$

The curve is a rectangular hyperbola; its semi-axes are each of length 1, and its eccentricity is therefore equal to $\sqrt{2}$.

EXERCISES

In Exercises 1–10 do not use transformation of coördinates. Determine the type of locus of each of the following equations. If the locus is non-degenerate, find its eccentricity.

1. $5x^2 - 8xy + 5y^2 - 12x + 6y + 9 = 0.$

2. $5x^2 - 8xy + 5y^2 - 12x + 6y + 10 = 0.$

3. $5x^2 - 8xy + 5y^2 - 12x + 6y = 0.$

4. $3y^2 - 4xy + 4x = 7.$

5. $4x^2 - 8xy + 4y^2 + 12x - 12y + 9 = 0.$

6. $9x^2 + 4xy + 6y^2 + 10x + 20y + 7 = 0.$

7. $x^2 - 4xy + y^2 + 6x - 6y + 36 = 0.$

8. $9x^2 - 24xy + 16y^2 + 6x + 242y = 0.$

9. $2x^2 + 5xy - 3y^2 - 3x + 5y - 2 = 0.$

10. $x^2 - 4xy + 4y^2 + 2x - 4y + 5 = 0.$

The following Exercises, No. 15 *excepted, refer to the general equation of second degree*

(A) $$ax^2 + 2bxy + cy^2 + 2dx + 2ey + f = 0.$$

11. Express in terms of invariants a condition that (A) represent a rectangular hyperbola.

12. In case (A) represents two parallel lines, find the distance between them in terms of invariants.

13. In case (A) represents a parabola, find the length of the latus rectum in terms of invariants.

14. In case (A) represents an ellipse, find the lengths of the axes in terms of invariants.

15. Prove that if

$$ax^2 + 2bxy + cy^2 + f = 0$$

is the equation of a hyperbola, then the equation of its asymptotes is

$$ax^2 + 2bxy + cy^2 = 0.$$

16. In case (A) represents a hyperbola, find $\tan \phi$ in terms of invariants, where ϕ is the angle between the asymptotes.

17. In case (A) represents a central conic, express the eccentricity e in terms of invariants.

18. Express in terms of invariants a condition that (A) represent a circle. When this condition is satisfied, express the radius in terms of invariants.

19. Prove by direct computation, using the equations of page 230, that the value of Δ is the same for (A) as it is when formed for any equation obtained from (A) by a translation of axes.

20. Prove by direct computation, using the equations of page 234, that the value of Δ is the same for (A) as it is when formed for any equation obtained from (A) by a rotation of axes.

113. The system $S_1 + kS_2 = 0$. If S_1 and S_2 are two expressions of the second degree in x and y, and k is a constant, then $S_1 + kS_2$ is of the second degree except in certain cases where all the terms of second degree cancel out. The equations $S_1 = 0$, $S_2 = 0$ represent two conics if each has a locus, and $S_1 + kS_2 = 0$ is also a conic (with the exceptions noted) provided it has any locus. Every solution, real or imaginary, of the simultaneous equations $S_1 = 0$, $S_2 = 0$ is a solution of $S_1 + kS_2 = 0$. This may be stated in geometric terms as follows, for the case of real intersections:

If $S_1 = 0$, $S_2 = 0$ *represent two conics with one or more points of intersection,* $S_1 + kS_2 = 0$ *is* (apart from exceptional cases) *a conic which passes through those points.*

If k is regarded as a parameter, $S_1 + kS_2 = 0$ is a **system of conics.**

Among the many applications of systems of this sort we shall here consider only two. The first is a method for solving simultaneous quadratic equations usually not given in elementary texts on algebra, the other is a way to find the equation of a conic passing through given points.

Let two simultaneous quadratic equations in x and y be $S_1 = 0$, $S_2 = 0$, or, written out,

(1) $$a_1x^2 + 2b_1xy + c_1y^2 + 2d_1x + 2e_1y + f_1 = 0,$$

(2) $$a_2x^2 + 2b_2xy + c_2y^2 + 2d_2x + 2e_2y + f_2 = 0.$$

If one of these equations, say (2), is degenerate, that is, if its left member is the product of real or imaginary factors of the first degree, we find these factors, equate them to zero, and solve each of the resulting equations simultaneously with (1) by methods given in algebra.

When neither (1) nor (2) is degenerate, let us use the property that every solution of the system (1), (2) is a solution of $S_1 + kS_2 = 0$ for every value of k, and let us choose k so that $S_1 + kS_2 = 0$ is degenerate. The necessary and

sufficient condition that the general equation of second degree be degenerate is that the invariant Δ be zero. Hence we choose k so as to satisfy the equation

$$(3) \qquad \begin{vmatrix} a_1 + ka_2 & b_1 + kb_2 & d_1 + kd_2 \\ b_1 + kb_2 & c_1 + kc_2 & e_1 + ke_2 \\ d_1 + kd_2 & e_1 + ke_2 & f_1 + kf_2 \end{vmatrix} = 0.$$

When the determinant has been expanded we see that equation (3) is of the form

$$(4) \qquad \Delta_2 k^3 + Ak^2 + Bk + \Delta_1 = 0,$$

where Δ_2 is the invariant Δ formed for equation (2), and Δ_1 is the invariant Δ formed for equation (1). Since we have supposed that (1) and (2) are not degenerate, so that $\Delta_1 \neq 0$, $\Delta_2 \neq 0$, equation (4) is of third degree and has at least one real root $k' \neq 0$. Then $S_1 + k'S_2 = 0$ can be factored *; if each factor is solved simultaneously with (1) or (2) we obtain all solutions of the system (1), (2). Incidentally we thus see that two simultaneous quadratic equations have not more than four solutions, unless the equations have a common factor that is not a constant.

Example 1. — Solve the simultaneous quadratic equations

$$2x^2 - xy - y^2 + x - 2y - 5 = 0 \qquad (S_1 = 0),$$
$$x^2 + y^2 = 5 \qquad (S_2 = 0).$$

Solution. — For this pair, equation (3) is

$$\begin{vmatrix} 2 + k & -\frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & -1 + k & -1 \\ \frac{1}{2} & -1 & -5 - 5k \end{vmatrix} = 0.$$

This reduces to

$$(2 + k)(5 - 5k^2) = 0.$$

The roots are $k = -2, -1, 1$.

* In the exceptional case where $S_1 + k'S_2$ reduces to a constant $\neq 0$, the system (1), (2) has no solution. If $S_1 + k'S_2 \equiv 0$, all solutions of $S_1 = 0$ are solutions of $S_2 = 0$.

The equation $S_1 + k'S_2 = 0$, with $k' = -2$, reduces to

$$(1 - y)(x + 3y + 5) = 0.$$

We solve the simultaneous systems

$$1 - y = 0, \quad S_2 = 0, \quad \text{and} \quad x + 3y + 5 = 0, \quad S_2 = 0,$$

to obtain solutions of the original pair $S_1 = 0$, $S_2 = 0$; these solutions are $(2, 1)$, $(-2, 1)$, $(-2, -1)$, $(1, -2)$. The corresponding points are the intersections of the hyperbola $S_1 = 0$ and the circle $S_2 = 0$.

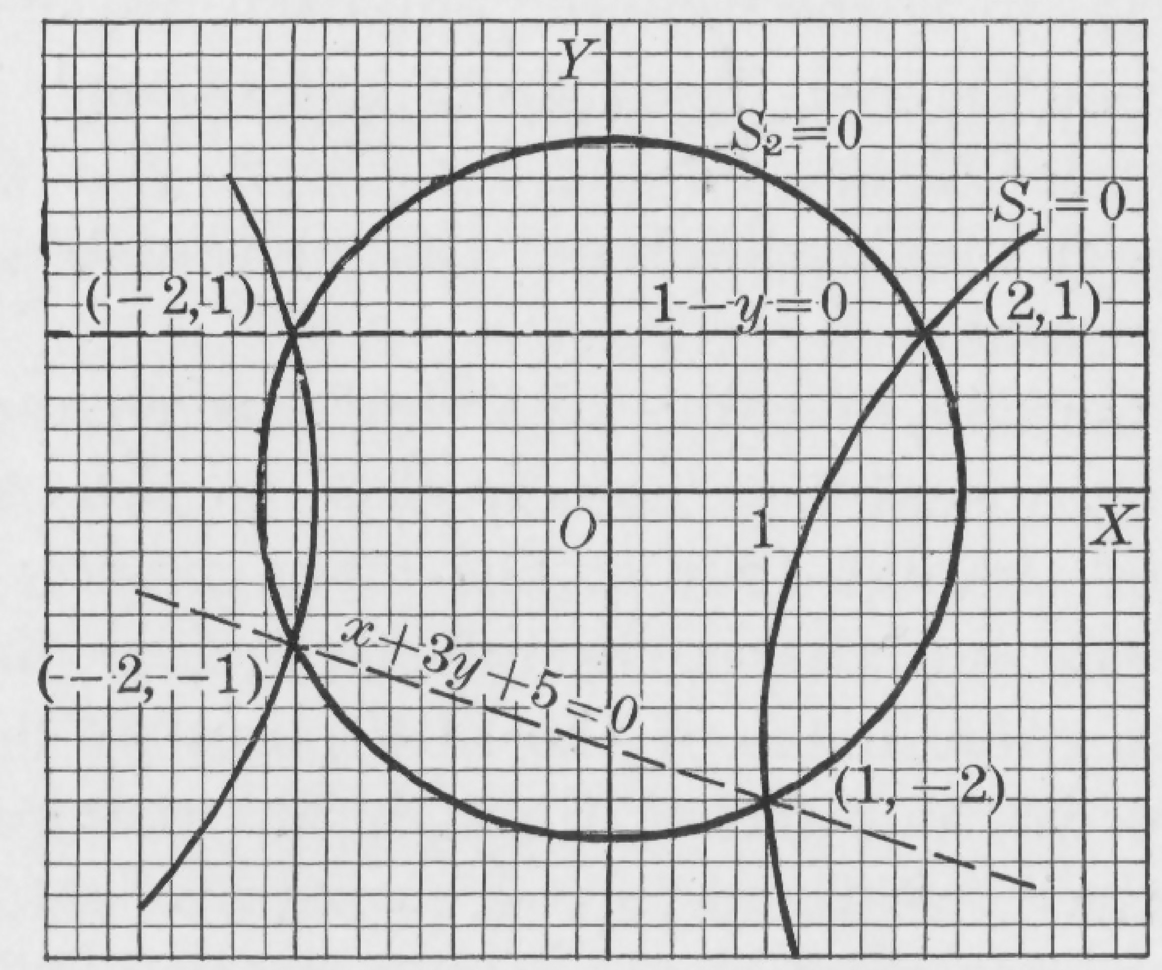


FIG. 118

We could also have obtained the solutions by finding the intersections of the two lines represented by $S_1 - 2S_2 = 0$ with the two represented by $S_1 - S_2 = 0$, or the two represented by $S_1 + S_2 = 0$.

The other problem we are to consider is that of determining an equation of a conic through given points. One method of finding the conic

$$(5) \qquad ax^2 + 2bxy + cy^2 + 2dx + 2ey + f = 0$$

passing through five given points (x_1, y_1), (x_2, y_2), (x_3, y_3), (x_4, y_4), (x_5, y_5) is to substitute these coördinates in (5), thus forming five equations which are to be solved for the ratios of a, b, c, d, e, f.

Another method gives directly as the solution of the problem the equation in determinant form

$$(6)\qquad \begin{vmatrix} x^2 & xy & y^2 & x & y & 1 \\ x_1^2 & x_1y_1 & y_1^2 & x_1 & y_1 & 1 \\ x_2^2 & x_2y_2 & y_2^2 & x_2 & y_2 & 1 \\ x_3^2 & x_3y_3 & y_3^2 & x_3 & y_3 & 1 \\ x_4^2 & x_4y_4 & y_4^2 & x_4 & y_4 & 1 \\ x_5^2 & x_5y_5 & y_5^2 & x_5 & y_5 & 1 \end{vmatrix} = 0.$$

To show that (6) is the required equation, we observe that it is of second degree, apart from exceptional cases, and that it is satisfied, for example, by $x = x_1$, $y = y_1$, because of the property of a determinant that it vanishes when two rows are the same.

Another method, usually involving less computation, is the following: Find two conics, $S_1 = 0$, $S_2 = 0$, each of which passes through four of the given points; then the conic $S_1 + kS_2 = 0$ passes through these four, and by substituting the coördinates of the fifth point we determine k so that $S_1 + kS_2 = 0$ passes through this point also.* We generally take $S_1 = 0$ and $S_2 = 0$ as pairs of straight lines, each pair passing through the four given points.

Example 2. — Find the equation of the conic which passes through the points $P_1(1, 0)$, $P_2(0, -1)$, $P_3(1, -1)$, $P_4(2, 2)$, $P_5(0, 0)$.

Solution. — Lines joining pairs of points are as follows:

P_2P_3: $y + 1 = 0$,
P_1P_4: $2x - y - 2 = 0$,
P_1P_3: $x - 1 = 0$,
P_2P_4: $3x - 2y - 2 = 0$.

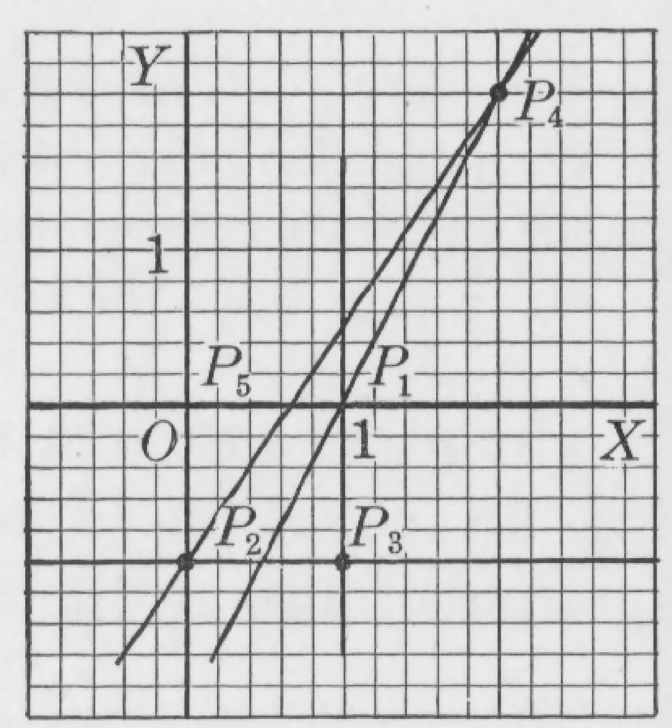


FIG. 119

The degenerate conics

$$(y + 1)(2x - y - 2) = 0, \quad (x - 1)(3x - 2y - 2) = 0$$

* If the fifth point is also on $S_2 = 0$, we obtain no solution for k, but $S_2 = 0$ is an answer to the problem.

have the four points of intersection P_1, P_2, P_3, P_4, and hence the conic

$$(7)\qquad (y + 1)(2x - y - 2) + k(x - 1)(3x - 2y - 2) = 0$$

passes through these points. We now choose k so that $P_5(0, 0)$ is also on the locus of (7). To do this we substitute $x = 0$, $y = 0$ in (7) and solve for k; we thus find that $k = 1$. If k is given this value, (7) is the equation of a conic through the five points specified. The equation, when simplified, is

$$3x^2 - y^2 - 3x - y = 0.$$

EXERCISES

Solve the following simultaneous quadratic equations. State which of the solutions are intersections of the corresponding conics.

1. $2x^2 - xy - y^2 - x - 2y - 12 = 0$,
$x^2 - y^2 = 4$.

2. $3x^2 - y^2 + 2y = 0$,
$x^2 + y^2 + 2x = 0$.

3. $3x^2 - xy + 8y^2 - 8x = 0$,
$x^2 + 4y^2 = 4x$.

4. $2x^2 - 2xy + 5y^2 - 4x = 0$,
$(x + y)^2 = 4x$.

Find an equation of each of the conics through the following points.

5. $(3, -2)$, $(1, 2)$, $(2, 1)$, $(-2, 3)$, $(-7, -2)$.

6. $(0, 0)$, $(0, 4)$, $(-3, 0)$, $(-8, -8)$, $(1, 4)$.

7. $(0, -2)$, $(-2, 0)$, $(2, 0)$, $(0, 2)$, $(-2, -2)$.

8. $(2, 2)$, $(0, 6)$, $(4, -2)$, $(4, 3)$, $(-2, 0)$.

9. $(-1, 0)$, $(2, -3)$, $(1, 2)$, $(2, 3)$, $(0, -1)$.

10. $(0, 0)$, $(1, -2)$, $(-1, -1)$, $(2, 4)$, $(2, 2)$.

11. Find equations of two parabolas through the points

$(1, 0)$, $(0, -2)$, $(1, -4)$, $(5, -2)$.

Hint. $b^2 - ac = 0$.

12. Find an equation of a rectangular hyperbola through the points

$(0, 1)$, $(1, 0)$, $(2, -2)$, $(3, 2)$.

Hint. $a + c = 0$.

CHAPTER XIII

CURVE FITTING

114. Introduction. In the preceding chapters we have considered curves which were completely defined by geometric properties or by equations. We now turn to a different type of problem.

In a multitude of applications of mathematics we obtain, by measuring, weighing, counting, or estimating, a set of corresponding pairs of values of two variables. Such a set may be given in the form of a table. It may also be represented graphically by a series of points in a plane. It is often an important problem to find a curve which will pass through or closely approach all of these points. This is called the problem of **fitting a curve** to the data.

For example, a wire under tension stretehes. We may measure, for a given wire, the stretch l for each of a number of tensions T. The results may be tabulated as follows, where T is the number of pounds, and l the number of thousandths of an inch:

TABLE I. *Tension T and stretch l of a wire*

$x =$	T	5	10	15	20	25	30	35
$y =$	l	3	6	8.5	11.5	15	17.5	21.5

The tabulated values are represented graphically by choosing perpendicular axes for T and l and plotting a point for each pair of values (Fig. 120). The points fall very nearly on a straight line. We wish to find the line which fits the points as well as possible.

The line thus determined may be used to estimate values

of l for values of T which are not given in the table. Even for a given value it is likely that the graph gives a better measure of l than was found by experiment, on account of errors of observation.

In the preceding example, the points lay near a straight line. Such is not always the case, however. When the empirical data are known, there arises first the question of the *type of curve* which we should fit to the data. This is frequently difficult to answer. The usual requirements for

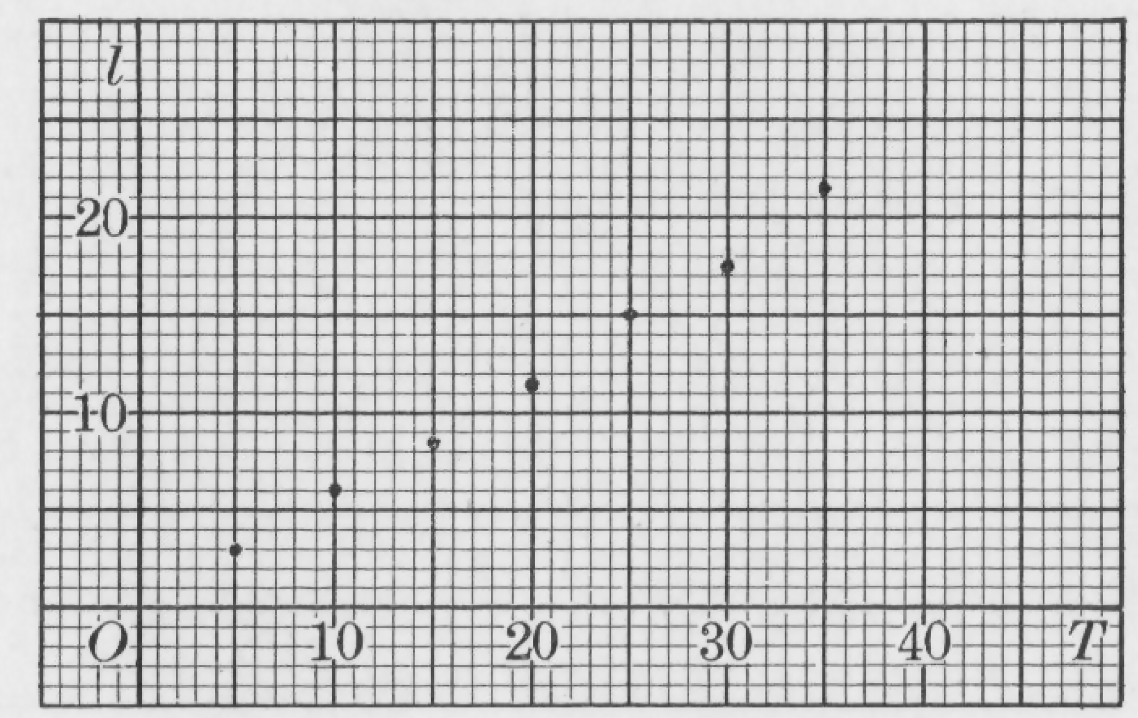


FIG. 120

practical purposes are that (a) the curve should accurately represent the trend of the points and (b) it should be as simple as possible to determine.

The most commonly used curves have equations of the following forms:

$$y = a + bx \quad \text{(straight line form),}$$
$$y = a + bx + cx^2 \quad \text{(parabolic form),}$$
$$y = ab^x \quad \text{(exponential form),}$$
$$y = ax^b \quad \text{(power form).}$$

The form to choose is often indicated by some theoretical consideration; otherwise we try a form which is suggested by inspection of the points to be fitted. The problem is to determine the constants in the equation of the chosen type so as to get as good a fit of the curve to the points as possible.

To solve this problem we must have some measure of goodness of fit which will permit us to determine the best fitting curve.

There are several methods in use for curve fitting, some of which will be described in the following paragraphs.

115. The method of average points. Suppose we wish to fit a straight line

$$l = a + bT$$

to the data of Table I, page 254. There are two constants, a and b, to be determined. It will suffice to find two points through which the line must pass.

By the **method of average points,** we first divide the set of points into two groups and find an average point for each group, — that is, one whose coördinates are the averages of the respective coördinates. We then find the line through the two average points. Thus we may take the first four points whose coördinates are given in Table I in one group, and the last three in another. The average point for the first group is (T_1, l_1), where

$$T_1 = \frac{5 + 10 + 15 + 20}{4} = 12.5,$$

$$l_1 = \frac{3 + 6 + 8.5 + 11.5}{4} = 7.25.$$

The average point for the second group is (T_2, l_2), where

$$T_2 = \frac{25 + 30 + 35}{3} = 30,$$

$$l_2 = \frac{15 + 17.5 + 21.5}{3} = 18.$$

The line through (T_1, l_1) and (T_2, l_2) has the equation

$$\frac{l - 7.25}{18 - 7.25} = \frac{T - 12.5}{30 - 12.5},$$

which simplifies to

$$(1) \qquad \begin{aligned} l &= -\frac{3}{7} + \frac{43}{70}T \\ &= -.429 + .614T. \end{aligned}$$

The preceding example illustrates the method of **average points,** applied to fitting a straight line. It is applicable for other types of curves. If the type of curve has, for example, three constants to be determined, we divide the points into

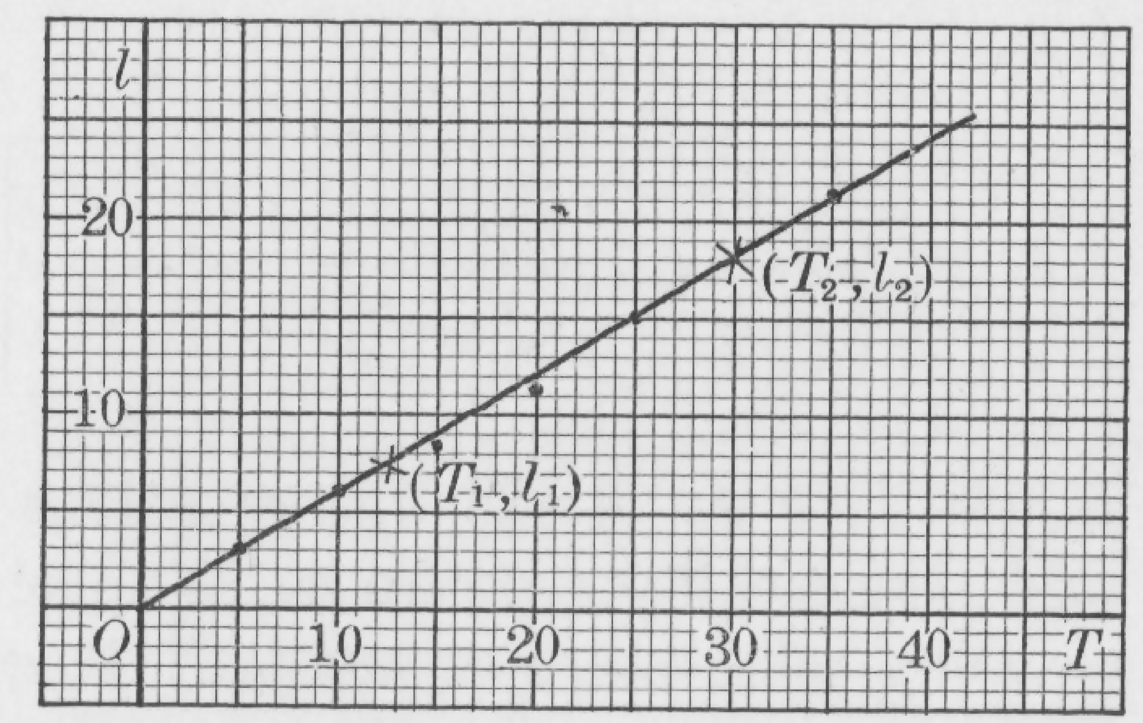


Fig. 121

three groups, find an average point for each group, substitute the coördinates of each of these points in the type equation, and solve for the three constants. It is to be noted that the choice of the points associated in groups is somewhat arbitrary, and that the resulting curve will usually depend upon the choice made. Hence there is not a uniquely determined curve even of a given type.

116. The method of average equations. Let us fit a parabola

$$(1) \qquad y = a + bx + cx^2$$

to the points given in Table II on the next page. If the curve actually passed through these points, the coördinates would satisfy the equation, and we would have

$$\begin{aligned} 5 &= a - 3b + 9c, \\ 2 &= a - 2b + 4c, \\ 0 &= a - b + c, \\ 3 &= a, \\ 4 &= a + b + c, \\ 17 &= a + 2b + 4c, \\ 25 &= a + 3b + 9c. \end{aligned}$$

These **observation equations,** as they are called, are, however, inconsistent unless the points all lie on a parabola of type (1).

TABLE II

x	-3	-2	-1	0	1	2	3
y	5	2	0	3	4	17	25

To determine a, b, c we divide the equations into three groups and add those of each group so as to obtain three equations; we then solve these equations. If we combine the first two of the observation equations, then the next three, then the last two, we obtain the set

$$\begin{aligned} 7 &= 2a - 5b + 13c, \\ 7 &= 3a \qquad\;\; + 2c, \\ 42 &= 2a + 5b + 13c. \end{aligned}$$

The solution of these equations is

$$a = 1.2, \qquad b = 3.5, \qquad c = 1.7.$$

Hence the equation of the parabola (Fig. 122) is

$$y = 1.2 + 3.5x + 1.7x^2.$$

The preceding example illustrates the fitting of a curve to a set of points by the method of average equations. A set of observation equations is obtained by substituting coördinates of given points in an equation containing constants to be determined. These equations are divided into groups and

added to obtain as many equations as there are unknowns. The resulting equations are solved for the unknowns. The work may become complicated if these equations are not linear.

By grouping the observation equations differently we may obtain different curves to fit a given set of points. Under

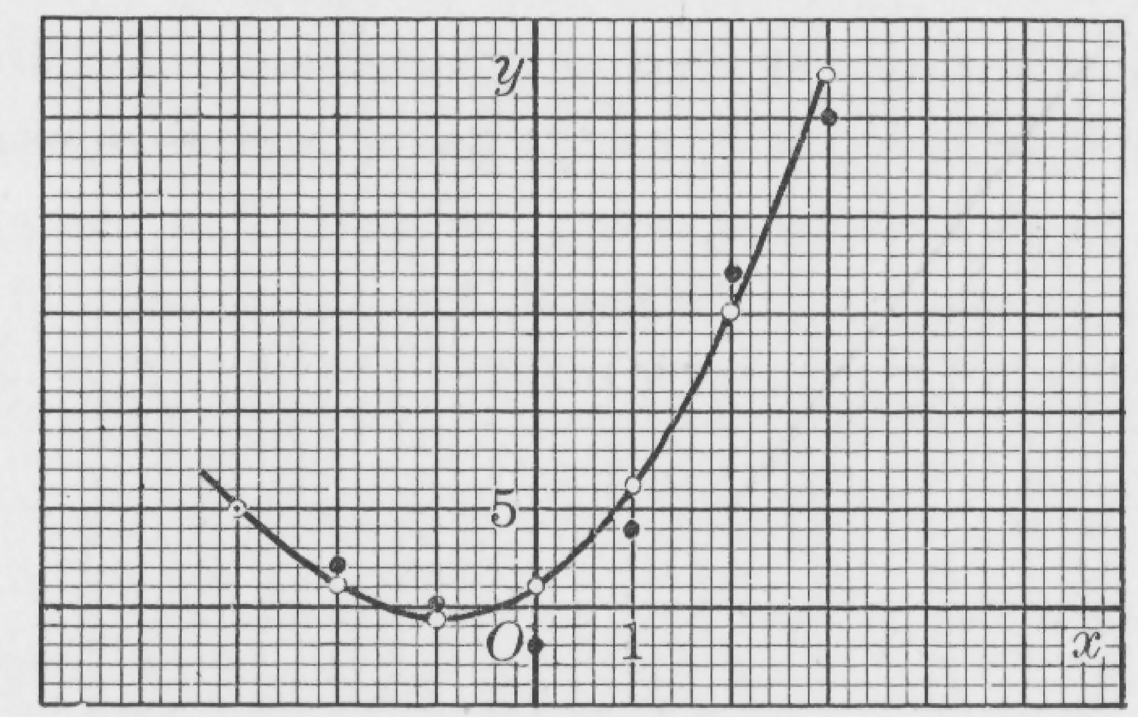


FIG. 122

some circumstances these curves may differ markedly. One must use care in selection of the groups in order to obtain a satisfactory curve.

EXERCISES

Find by the method of average points a straight line fitting the points in each of the following Exercises 1–2. Plot the given points, the average points, and the line.

1.

x	5	8	11	14	17	20	23	26	29	32
y	11	9	10	8	7	5	5	5	6	6

2.

x	60	64	68	72	76	80	84	88
y	22	24	27	28	30	31	32	33

Hint. In Exercise 2 change the origin to $(70, 30)$ and let $x' = 4w$; determine the equation $w = a + by'$ which fits the points; then return to the variables x and y.

3. Proceed as in Exercise 1, using the method of average equations.

4. Proceed as in Exercise 2, using the method of average equations.

5. Find by the method of average points a parabola of the type $y = a + bx + cx^2$ fitting the points of Exercise 1. Plot the points, the average points, and the parabola.

Hint. Change the origin to the point (17, 7) and let $x' = 3w$; determine the equation $y' = a + bw + cw^2$ which fits the points; then return to the variables x and y.

6. Proceed as in Exercise 5, using the points of Exercise 2.

7. Proceed as in Exercise 5, using the method of average equations.

8. Proceed as in Exercise 6, using the method of average equations.

9. In an experiment with falling bodies the distance d cm. through which a body fell in t seconds was observed to be as given in the following table:

t	0	1	2	3	4
d	0	7.51	28.55	62.77	116.40

Assume that there is a relation of the form $d = at^2$ between d and t. (*a*) By the method of average points find the value of a, and use the resulting formula to determine how far the body fell in 1, 2, 3, and 4 seconds. (*b*) By the method of average equations find the value of a, and use the resulting formula to compute how far the body fell in 1, 2, 3, and 4 seconds. (*c*) Compare the results of (*a*) and (*b*) with the observed values of d. Which method gave the better value of a?

117. The method of least squares. This method requires a little more computation than is involved in the preceding methods, but it gives more reliable results.

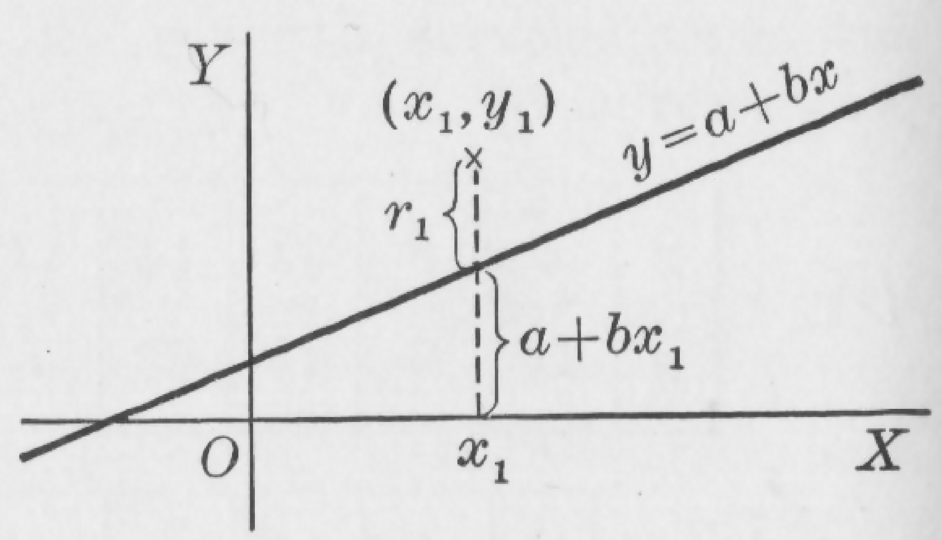


FIG. 123

Let $(x_1, y_1), (x_2, y_2), \cdots,$ be a set of points to which we wish to fit a curve. Suppose first that the curve is of the straight line type

$$y = a + bx.$$

For a given value of x, say $x = x_1$, we obtain from the equation an estimated value of y, namely $a + bx_1$. The **residual,** r_1, of this value is

$$r_1 = y_1 - (a + bx_1).$$

There is for each point $(x_1, y_1), (x_2, y_2), \cdots$, a residual. Let S^2 be the average of the squares of the residuals; thus we have

$$S^2 = \frac{1}{n}(r_1^2 + r_2^2 + \cdots + r_n^2) = \frac{1}{n}\Sigma r^2,$$

where Σr^2 means the sum of the squares of the r's. Each r, and hence S^2, depends upon the choice of a and b. *The criterion of least squares requires that* a *and* b *be chosen so that* S² *is made as small as possible.*

If the type curve is a parabola

$$y = a + bx + cx^2,$$

the *residuals* are

$$r_1 = y_1 - (a + bx_1 + cx_1^2), \quad r_2 = y_2 - (a + bx_2 + cx_2^2), \cdots.$$

We define S^2 as before, and the criterion of least squares requires that a, b, and c be chosen so that S^2 is a minimum.

It should be clear from these examples how the criterion of least squares applies to any type curve. An equation of the type curve contains parameters, $a, b, c, \cdots$. For each choice of these parameters, there is a residual for each of the points (x_1, y_1) $(x_2, y_2), \cdots$. The problem is to determine the values of the parameters which will make the sum of the squares of the residuals a minimum.

In order to show how this criterion leads to equations from which the constants (a and b in the case of the straight line, or a, b, and c in that of the parabola) are determined we need a theorem which we develop in the following section. The details of the method of least squares will be shown in later sections.

118. The minimum of a quadratic function. For the quadratic expression $Au^2 + 2Bu + C$, where u is the variable, we have

$$Au^2 + 2Bu + C = \frac{1}{A}(A^2u^2 + 2ABu + AC)$$
$$= \frac{1}{A}[(Au + B)^2 + (AC - B^2)].$$

Since $(Au + B)^2$ is positive or zero and $AC - B^2$ does not depend on u, we have the following theorem.

Theorem. *The minimum value of* $Au^2 + 2Bu + C$, *where* A *is positive, occurs when* $Au + B = 0$; *the minimum value is* $(AC - B^2)/A$.

FITTING A LINE OF THE TYPE $y = a$ *BY LEAST SQUARES*

119. The arithmetic mean as best value. Let us first consider the very simple case of fitting a line which is parallel to the x-axis; the equation has the form

(1) $y = a,$

where a is a constant.

The residuals in this case for points (x_1, y_1), (x_2, y_2), $\cdots$, are

$r_1 = y_1 - a,$

$r_2 = y_2 - a, \cdots.$

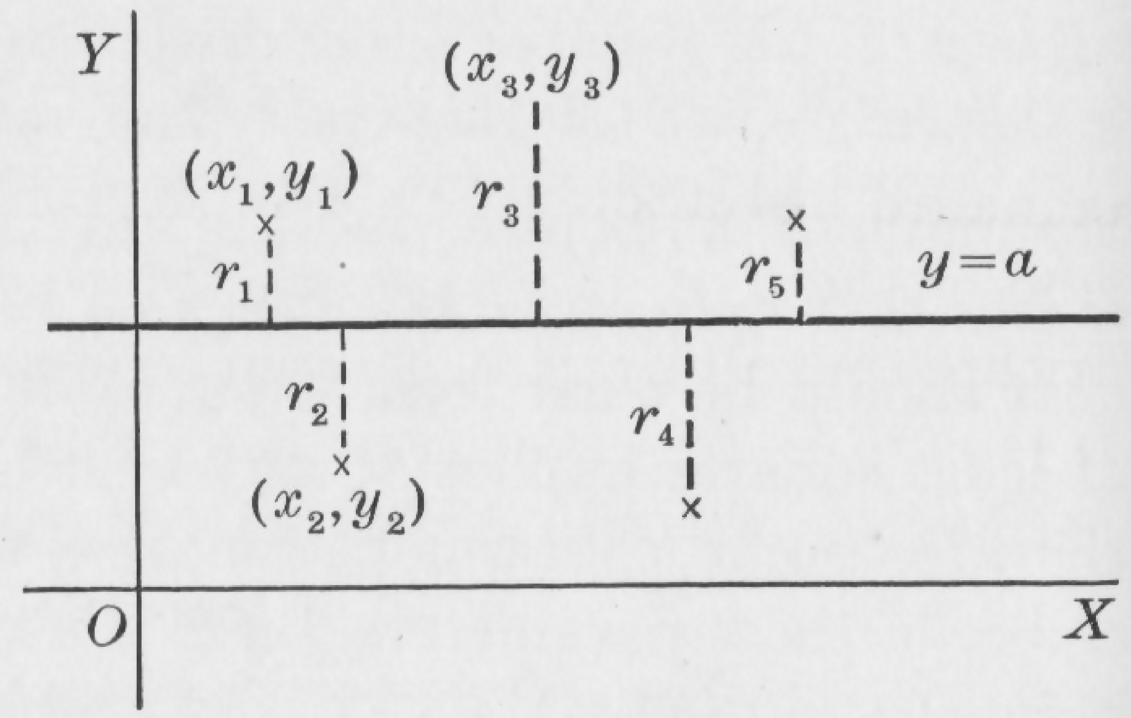


FIG. 124

Hence, in the notation of § 117, if S^2 is the average of the squares of the residuals, we have

$$nS^2 = (y_1 - a)^2 + (y_2 - a)^2 + \cdots + (y_n - a)^2$$
$$(2) \qquad = na^2 - 2a(y_1 + y_2 + \cdots + y_n) + (y_1^2 + y_2^2 + \cdots + y_n^2)$$
$$= na^2 - 2a\Sigma y + \Sigma y^2.$$

We apply the theorem of § 118 to find the value of a which makes this quadratic expression in a a minimum. Here

$$u = a, \qquad A = n, \qquad B = -\Sigma y, \qquad C = \Sigma y^2;$$

hence for the minimum value of nS^2, and thus of S^2, we must choose a so that $na - \Sigma y = 0$, that is,

$$(3) \qquad a = \frac{\Sigma y}{n} = \frac{y_1 + y_2 + \cdots + y_n}{n}.$$

Thus *a is the arithmetic mean of the* y's for the line of type (1) which fits best according to the criterion of least squares.

From the theorem of § 118, we find, after dividing formula (2) of the present section by n, that the minimum value of S^2, which we shall denote by σ^2, is

$$(4) \qquad \sigma^2 = \frac{\Sigma y^2}{n} - \left(\frac{\Sigma y}{n}\right)^2 = \frac{\Sigma y^2}{n} - a^2.$$

We note that σ^2, from its definition, *is the average of the squares of the residuals* (*or* **deviations**) *of the* y's *from the arithmetic mean of the* y's. The number σ is called the **standard deviation** of the y's. It is a measure of the dispersion of the y's. If σ is small, the y's tend to lie close to the arithmetic mean; if it is large they are more widely spread.

It is to be observed that the values x_1, x_2, $\cdots$, played no part in the preceding discussion. The results are therefore applicable to the problem of finding a single number a which best represents a series of values of a single variable y by the criterion of least squares. The arithmetic mean of the y's is the required number.

120. Computation of the arithmetic mean and standard deviation. The formulas (3) and (4) of § 119 may be used to compute the arithmetic mean a and standard deviation σ of a set of y's, but if the y's are large numbers, the labor is reduced by the following device.

Let A be a number near the arithmetic mean; we call A an **assumed mean.** Let

$$(1) \qquad y_1 = A + y_1', \qquad y_2 = A + y_2', \cdots;$$

then

$$y_1 + y_2 + \cdots + y_n = nA + (y_1' + y_2' + \cdots + y_n').$$

Hence the mean of the y's is given by the formula

$$(2) \qquad a = A + \frac{\Sigma y'}{n} = A + a',$$

where

$$(3) \qquad a' = \frac{\Sigma y'}{n}.$$

On squaring the equations (1) we obtain

$$y_1^2 = A^2 + 2Ay_1' + y_1'^2,$$
$$y_2^2 = A^2 + 2Ay_2' + y_2'^2,$$
$$\cdots\cdots\cdots\cdots$$

When we add these equations and divide by n, we get

$$\frac{\Sigma y^2}{n} = A^2 + 2A\frac{\Sigma y'}{n} + \frac{\Sigma y'^2}{n}.$$

From (2) we have

$$a^2 = A^2 + 2A\frac{\Sigma y'}{n} + \left(\frac{\Sigma y'}{n}\right)^2.$$

By subtracting the latter equation from the preceding one it now follows from (4), § 119, that

$$(4) \qquad \sigma^2 = \frac{\Sigma y'^2}{n} - \left(\frac{\Sigma y'}{n}\right)^2 = \frac{\Sigma y'^2}{n} - a'^2.$$

Example. — Find the arithmetic mean and standard deviation of the set of numbers 76, 78, 79, 80, 81, 82, 84, 87.

Solution. — If we use formula (3), § 119, we have

$$a = \frac{76 + 78 + \cdots + 87}{8} = \frac{647}{8} = 80.875.$$

By formula (4), § 119, we have

$$\sigma^2 = \frac{76^2 + 78^2 + \cdots + 87^2}{8} - 80.875^2.$$

To carry through this computation is somewhat laborious, even with a table of squares at hand.

Let us take as an assumed mean $A = 80$, and use formulas (2), (3), and (4) of the present section. The work is carried through most readily in tabular form as shown below.

y	y'	y'^2
87	7	49
84	4	16
82	2	4
81	1	1
80	0	0
79	-1	1
78	-2	4
76	-4	16
Σ	7	91

We have

$$a' = \tfrac{7}{8} = .875,$$

$$\frac{\Sigma y'^2}{n} = 11.375.$$

Hence

$$a = 80 + .875 = 80.875,$$
$$\sigma^2 = 11.375 - .875^2$$
$$= 10.6094,$$
$$\sigma = 3.26.$$

EXERCISES

The following Exercises 1–6 *refer to Table I, page* 266. *In each Exercise plot the values of the variable along a line, calculate the arithmetic mean* a *and standard deviation* σ, *and plot the points* a, a + σ, *and* a − σ.

1. (*a*) Grades given by Instructor A. (*b*) Grades given to Paper I.

2. (*a*) Grades given by Instructor B. (*b*) Grades given to Paper II.

3. (*a*) Grades given by Instructor C. (*b*) Grades given to Paper III.

4. (*a*) Grades given by Instructor D. (*b*) Grades given to Paper IV.

5. (*a*) Grades given by Instructor E. (*b*) Grades given to Paper V.

6. (*a*) Grades given by Instructor K. (*b*) Grades given to Paper X.

TABLE I

Grades by ten instructors on algebra test papers, each grade based on 25 answers (Northwestern University, 1920)

Papers	A	B	C	D	E	F	G	H	J	K
I	90	90	89	90	91	92	89	92	91	86
II	83	82	82	81	82	83	84	85	83	85
III	84	78	81	77	78	75	79	83	85	77
IV	74	73	71	74	72	72	70	71	77	65
V	67	68	65	63	67	67	65	66	66	63
VI	58	59	61	61	61	59	58	63	66	48
VII	55	58	52	55	57	57	55	52	61	51
VIII	50	50	46	48	44	49	47	51	60	42
IX	35	43	38	38	40	41	40	52	28	24
X	23	25	22	23	26	28	24	27	28	15

7. If in a series of values of x, the value x_1 occurs f_1 times, the value x_2 occurs f_2 times, $\cdots$, show that the value of the arithmetic mean a of the x's is given by the formula

$$a = \frac{\Sigma fx}{\Sigma f} = \frac{\Sigma fx'}{\Sigma f} + a',$$

and the standard deviation σ by

$$\sigma^2 = \frac{\Sigma fx'^2}{\Sigma f} - a'^2.$$

8. Use the formulas of Exercise 7 to find the arithmetic mean and standard deviation of the heights of students which are given in the following table, where h denotes height in inches, and f is the number of students of the corresponding height.

TABLE II

Heights of students

h	60.5	62.0	63.5	65.0	66.5	68.0	69.5	71.0	72.5	74.0	75.5
f	1	3	14	32	61	80	71	35	24	2	1

FITTING A LINE OF THE TYPE y = a + bx *BY LEAST SQUARES*

121. The normal equations. In the notation of § 117, we are to minimize the value of S^2, where

$$nS^2 = \Sigma r^2,$$

and

$$r_1 = y_1 - (a + bx_1), \quad r_2 = y_2 - (a + bx_2), \cdots.$$

We find that

$$\begin{aligned} r_1^2 &= [(a + bx_1) - y_1]^2 \\ &= a^2 + 2abx_1 + b^2x_1^2 - 2ay_1 - 2bx_1y_1 + y_1^2, \\ r_2^2 &= a^2 + 2abx_2 + b^2x_2^2 - 2ay_2 - 2bx_2y_2 + y_2^2, \\ &\cdots\cdots\cdots\cdots, \end{aligned}$$

and hence, by adding these n equations, we have

$$nS^2 = na^2 + 2ab\Sigma x + b^2\Sigma x^2 - 2a\Sigma y - 2b\Sigma xy + \Sigma y^2.$$

This is a quadratic expression in a; to get its minimum value, a must be chosen so that (Theorem, § 118)

$$na + b\Sigma x - \Sigma y = 0.$$

The expression for nS^2 is also a quadratic in b; hence b must be chosen so that

$$b\Sigma x^2 + a\Sigma x - \Sigma xy = 0.$$

These last two equations may be written

$$\text{(1)} \qquad a\Sigma 1 + b\Sigma x - \Sigma y = 0,$$
$$\text{(2)} \qquad a\Sigma x + b\Sigma x^2 - \Sigma xy = 0.$$

They are known as the **normal equations.** We solve them for a and b, and substitute in the equation

$$\text{(3)} \qquad y = a + bx$$

to get the straight line of best fit. The line is sometimes called the **line of regression of y on x.**

Equation (1) is obtained from (3) if x and y in the latter

are replaced by the average x and the average y respectively. It follows that *the line of regression passes through the average point.*

122. Aids to computation. The computation involved in finding equations (1) and (2) of § 121, and solving for a and b, is often very laborious. It is important to find short cuts where this is possible. A marked simplification is often made by a change of variables from x and y to x' and y' by the substitution

$$x = h + ux', \qquad y = k + vy', \tag{1}$$

provided h, k, u, and v are properly chosen constants. This substitution may be regarded as a translation of axes to a new origin at (h, k) and a change of units on the axes.

The equation of the line

$$y = a + bx \tag{2}$$

in the new variables becomes, by (1),

$$y' = A + Bx', \tag{3}$$

where

$$A = \frac{a + bh - k}{v}, \qquad B = \frac{bu}{v}. \tag{4}$$

If we solve (4) for a and b we have

$$a = k + vA - h\frac{Bv}{u}, \qquad b = \frac{Bv}{u}. \tag{5}$$

The normal equations (1), (2) of § 121 may be expressed in terms of A, B, x', y'. We have

$$\begin{aligned} x_1 &= h + ux_1', & y_1 &= k + vy_1', \\ x_2 &= h + ux_2', & y_2 &= k + vy_2', \\ &\cdots\cdots & &\cdots\cdots \end{aligned}$$

Hence

$$\begin{aligned} \Sigma x &= nh + u\Sigma x', \qquad \Sigma y = nk + v\Sigma y', \\ \Sigma x^2 &= nh^2 + 2hu\Sigma x' + u^2\Sigma x'^2, \\ \Sigma xy &= nhk + ku\Sigma x' + hv\Sigma y' + uv\Sigma x'y'. \end{aligned}$$

On substituting these expressions in (1), (2) of § 121, and noting that $n = \Sigma 1$, we obtain

$$\begin{aligned} &(a + bh - k)\Sigma 1 + bu\Sigma x' - v\Sigma y' = 0, \\ &(ah + bh^2 - hk)\Sigma 1 + (au + 2bhu - ku)\Sigma x' - hv\Sigma y' \\ &\qquad\qquad + bu^2\Sigma x'^2 - uv\Sigma x'y' = 0. \end{aligned}$$

Multiply the first of these equations by h and subtract from the second, then divide by uv; the result is

$$\frac{a + bh - k}{v}\Sigma x' + \frac{bu}{v}\Sigma x'^2 - \Sigma x'y' = 0.$$

In view of (4), the first of the preceding equations and this one may be written respectively

$$A\Sigma 1 + B\Sigma x' - \Sigma y' = 0, \tag{6}$$

$$A\Sigma x' + B\Sigma x'^2 - \Sigma x'y' = 0. \tag{7}$$

These are the normal equations expressed in terms of the new letters. As we have remarked, they are often simpler than equations (1) and (2) of § 121.

Example. — Fit a line to the data of Table I, page 254.

Solution. — We choose h and k near the arithmetic means of the values of T and l, $h = 20$, $k = 12$. It will simplify calculations to take $u = 5$, $v = .5$, in the substitution $T = h + uT'$, $l = k + vl'$, so that

$$T' = \frac{T - 20}{5}, \qquad l' = 2(l - k).$$

The calculation of the coefficients in equations (6) and (7) is conveniently carried out as shown in the accompanying table.

T	l	T'	l'	T'^2	$T'l'$
35	21.5	3	19	9	57
30	17.5	2	11	4	22
25	15	1	6	1	6
20	11.5	0	− 1	0	0
15	8.5	− 1	− 7	1	7
10	6	− 2	− 12	4	24
5	3	− 3	− 18	9	54
		0	− 2	28	170

Thus we have

$$\Sigma 1 = 7, \quad \Sigma T' = 0, \quad \Sigma l' = -2, \quad \Sigma T'^2 = 28, \quad \Sigma T'l' = 170.$$

Equations (6) and (7) become

$$7A + 2 = 0,$$
$$28B - 170 = 0.$$

Hence

$$A = -\frac{2}{7}, \qquad B = \frac{85}{14}, \qquad l' = -\frac{2}{7} + \frac{85}{14}T';$$

and, from (5),

$$a = 12 - \frac{1}{7} - \frac{85}{7} = -\frac{2}{7}, \qquad b = \frac{17}{28},$$

so that

$$(8) \qquad l = -\frac{2}{7} + \frac{17}{28}T = -.286 + .607T.$$

This agrees fairly well with the equation found in § 115 by the method of average points.

EXERCISES

Find by the Method of Least Squares the line of best fit if x *and* y *are chosen as stated in each of Exercises 1–7. Plot the points and the line.*

1. As in Exercise 1, page 259.

2. As in Exercise 2, page 259.

3. Take x as a grade by Instructor A, and y the corresponding grade by instructor B in Table I, page 266.

4. Take x as a grade by instructor D, and y the corresponding grade by instructor E in Table I, page 266.

5. An instructor graded the same algebra test papers in 1920 and again in 1922. The corresponding grades are shown in the following table:

Grades in 1920	86	85	77	65	63	48	51	42	24	15
Grades in 1922	83	77	72	64	57	47	48	41	25	10

Let x be the grades in 1920 and y those in 1922.

6. A student measured the voltage x and the corresponding amperage y of a Mazda lamp as follows:

x	10	20	30	40	50	60	70	80	90	100	110
y	.31	.38	.45	.52	.59	.65	.69	.74	.78	.82	.86

7. In another experiment like that of Exercise 6 results were as follows:

x	10	20	30	40	50	60	70	80	90	100	110
y	.06	.10	.17	.23	.29	.37	.43	.50	.57	.65	.72

8. In the normal equations (1), (2), § 121, let $\bar{x}$ and $\bar{y}$ be the arithmetic means of the x's and y's, and let

$$\sigma_x^2 = \frac{\Sigma x^2}{n} - \bar{x}^2, \qquad p = \frac{\Sigma xy}{n} - \bar{x}\bar{y}, \qquad \sigma_y^2 = \frac{\Sigma y^2}{n} - \bar{y}^2.$$

Show that the equation of the line of best fit may be written

$$y - \bar{y} = \frac{p}{\sigma_x^2}(x - \bar{x}).$$

Let

$$r = \frac{p}{\sigma_x \sigma_y}$$

and show that the preceding equation may be written

$$\frac{y - \bar{y}}{\sigma_y} = r\frac{x - \bar{x}}{\sigma_x}.$$

(The line is often called the *line of regression* of y on x; and r is called the *Pearson coefficient of correlation* of y and x.)

9. In Exercise 8, substitute

$$x = h + ux', \quad y = k + vy',$$

and show that if

$$\bar{x}' = \frac{\Sigma x'}{n}, \qquad \bar{y}' = \frac{\Sigma y'}{n},$$

$$\sigma_{x'}^2 = \frac{\Sigma x'^2}{n} - \bar{x}'^2, \quad \sigma_{y'}^2 = \frac{\Sigma y'^2}{n} - \bar{y}'^2, \quad p' = \frac{\Sigma x'y'}{n} - \bar{x}'\bar{y}',$$

then

$$\bar{x} = h + u\bar{x}', \qquad \bar{y} = k + v\bar{y}',$$

$$\sigma_x^2 = u^2\sigma_{x'}^2, \quad \sigma_y^2 = v^2\sigma_{y'}^2, \quad p = uvp', \quad r = \frac{p'}{\sigma_{x'}\sigma_{y'}}.$$

[By proper choice of assumed means h and k and of factors u and v, the

substitution may replace large values x and y by small values x' and y'; the calculation of $\bar{x}$, $\bar{y}$, σ_x, σ_y, p and r is thus simplified by first calculating $\bar{x}'$, $\bar{y}'$, $\sigma_{x'}$, $\sigma_{y'}$, and p', and then using the final formulas above.]

10. By calculating $\bar{x}$, $\bar{y}$, σ_x, σ_y, p, r, by the method suggested in Exercise 9, and using the result of Exercise 8, find the line of best fit (i.e., of regression of y on x) for the data of Exercise 6.

11. Proceed as in Exercise 10 using the data of Exercise 7.

12. By use of the normal equations (1), (2) of § 121, show that the expression for nS^2, § 121, simplifies to

$$nS^2 = -a\Sigma y - b\Sigma xy + \Sigma y^2.$$

Show that, in the notation of Exercise 8, it follows that

$$S^2 = \sigma_y^2(1 - r^2).$$

13. Using results of Exercise 12, show that the points to which a line is being fitted lie on a straight line if and only if $r = \pm 1$, and that for a given standard deviation σ_y of the y's, the line fits most poorly when $r = 0$. (In view of these facts S/σ_y and r are both used to measure closeness of fit of the line to the points.)

14. Use results of Exercise 12 to find the value of S^2 for the data of Exercise 6.

FITTING A PARABOLA OF THE TYPE $y = a + bx + cx^2$ *BY LEAST SQUARES*

123. The normal equations. The discussion for the parabola is very similar to that for a straight line as given in §§ 121, 122. The resulting normal equations are found to be

$$\begin{aligned} a\Sigma 1 + b\Sigma x + c\Sigma x^2 - \Sigma y &= 0, \\ a\Sigma x + b\Sigma x^2 + c\Sigma x^3 - \Sigma xy &= 0, \\ a\Sigma x^2 + b\Sigma x^3 + c\Sigma x^4 - \Sigma x^2 y &= 0. \end{aligned} \tag{1}$$

From these equations we determine a, b, and c, and have the parabola of best fit

$$y = a + bx + cx^2. \tag{2}$$

It may simplify the computation to translate the axes and take new units on the axes by the substitution

$$x = h + ux', \qquad y = k + vy'. \tag{3}$$

If the equation of the curve becomes in the new variables

$$y' = A + Bx' + Cx'^2, \tag{4}$$

the normal equations become

$$\begin{aligned} A\Sigma 1 + B\Sigma x' + C\Sigma x'^2 - \Sigma y' &= 0, \\ A\Sigma x' + B\Sigma x'^2 + C\Sigma x'^3 - \Sigma x'y' &= 0, \\ A\Sigma x'^2 + B\Sigma x'^3 + C\Sigma x'^4 - \Sigma x'^2 y' &= 0, \end{aligned} \tag{5}$$

from which A, B, and C are determined. From equation (4) we obtain (2) by using the inverse of the substitution (3),

$$x' = \frac{x - h}{u}, \qquad y' = \frac{y - k}{v}.$$

The generalization of the preceding equations to the case of a curve of higher degree is fairly obvious.

Example. — Fit a parabola $y = a + bx + cx^2$ to the points $(-4, 3)$, $(0, 9)$, $(4, 9)$, $(8, 12)$, $(12, 9)$, $(16, 6)$.

Solution. — The point $(4, 9)$ has a central location; we translate axes to this point as a new origin. The values of x have a factor 4, those of y a factor 3, and we change units accordingly. We substitute

$$x = 4 + 4x', \qquad y = 9 + 3y'.$$

The x', y' coördinates of the points therefore are

$$(-2, -2), (-1, 0), (0, 0), (1, 1), (2, 0), (3, -1).$$

We now have

$\Sigma 1 = 6$, the number of points;
$\Sigma x' = 3$, the sum of the x' coördinates;
$\Sigma x'^2 = 19, \quad \Sigma x'^3 = 27, \quad \Sigma x'^4 = 115,$
$\Sigma y' = -2, \quad \Sigma x'y' = 2, \quad \Sigma x'^2 y' = -16.$

Hence the normal equations (5) are

$$\begin{aligned} 6A + 3B + 19C + 2 &= 0, \\ 3A + 19B + 27C - 2 &= 0, \\ 19A + 27B + 115C + 16 &= 0. \end{aligned}$$

The solution of these equations is

$$A = \frac{17}{35} = 0.4857, \quad B = \frac{143}{280} = 0.5107, \quad C = -\frac{19}{56} = -0.3393.$$

The equation of the required parabola becomes

$$y' = 0.4857 + 0.5107x' - 0.3393x'^2.$$

On substituting

$$y' = \frac{y-9}{3}, \quad x' = \frac{x-4}{4},$$

and simplifying, we obtain

$$y = 7.907 + 0.892x - 0.0636x^2.$$

The graph is shown in Figure 125.

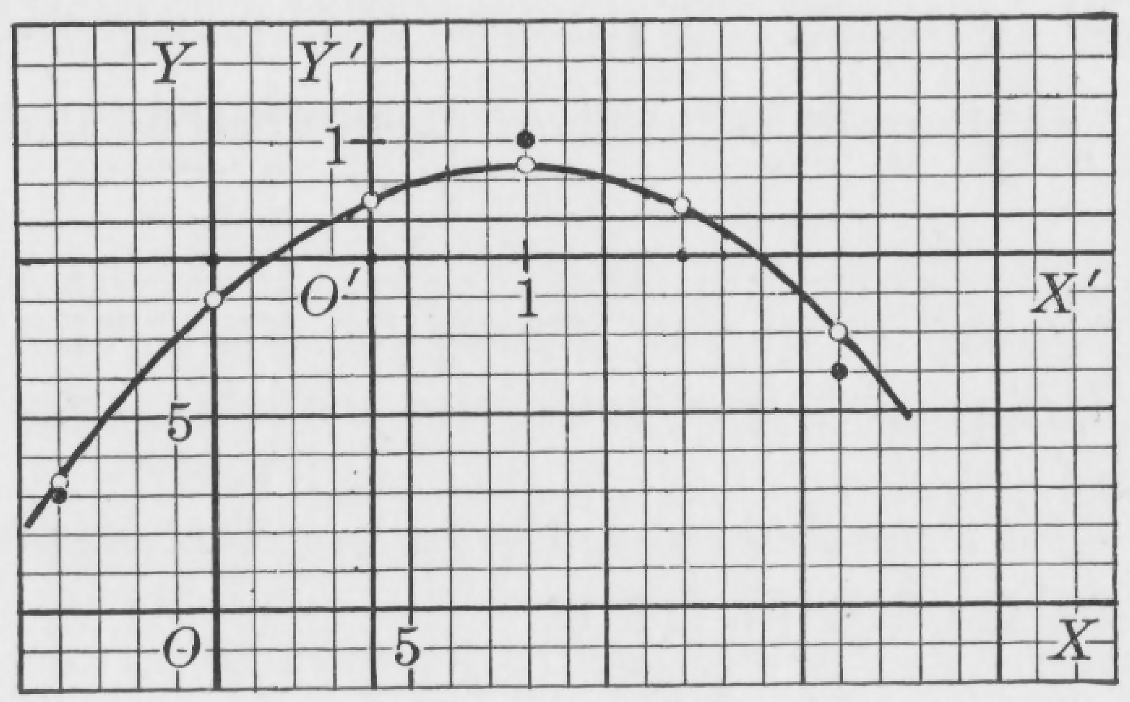


FIG. 125

EXERCISES

Solve by the Method of Least Squares.

1. Fit a parabola of the type $d = a + bt + ct^2$ to the points of Exercise 9, page 260. Draw the locus.

2. Fit a parabola of the type $y = a + bx + cx^2$ to the points of Exercise 2, page 259. Draw the curve.

3. Proceed as in Exercise 2 with the points of Table II, page 258.

4. Fit a curve of the type $y = a + bx + cx^2 + dx^3$ to the points of the Example, § 123. Compare with the parabola shown in Figure 125. Do you consider the cubic curve better than the parabola? Why?

5. Fit a parabola of the type $y = a + bx^2$ to the points whose coördinates are as follows:

x	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8
y	2.64	2.63	2.60	2.58	2.56	2.52	2.49	2.43

FITTING POWER AND EXPONENTIAL FUNCTIONS

124. The power function. A function of the type ax^b, where a and b are constants, will be called a **power function.** The constant b may be positive or negative, integral, fractional, or irrational. Let us consider how a curve of the type

(1) $$y = ax^b$$

may be fitted to a set of points, by choice of a and b.

The methods of the preceding sections are not easily applied directly. A device which is satisfactory is the following. Take the logarithm * of each member of equation (1). We then have

(2) $$\log y = \log a + b \log x.$$

Let

(3) $$Y = \log y, \quad X = \log x, \quad A = \log a;$$

the preceding equation becomes

(4) $$Y = A + bX.$$

Replace the points $(x_1, y_1), (x_2, y_2), \cdots$, by $(X_1, Y_1), (X_2, Y_2), \cdots$, where

$$X_1 = \log x_1, \quad Y_1 = \log y_1, \quad X_2 = \log x_2, \quad Y_2 = \log y_2, \cdots.$$

Let us now fit the line (4) to the points $(X_1, Y_1), (X_2, Y_2), \cdots$, by one of the methods of the preceding sections, thus determining A and b. When a is determined from the relation $A = \log a$, the problem is solved.

It is clear from (3) that we can employ this method only when a, x, and y are positive, since there is no real logarithm of a negative number or zero. In case the x or y coördinate of some point is negative it may be possible to translate the axes and find an equation of the type (1) for the new axes. The method is illustrated in the Example which is solved on pages 277, 278.

* In this section, and in those that follow, all logarithms are taken to the base 10.

In Figure 126 are shown graphs of curves of type (1); for these curves $a = 1$, and the values of b are indicated.

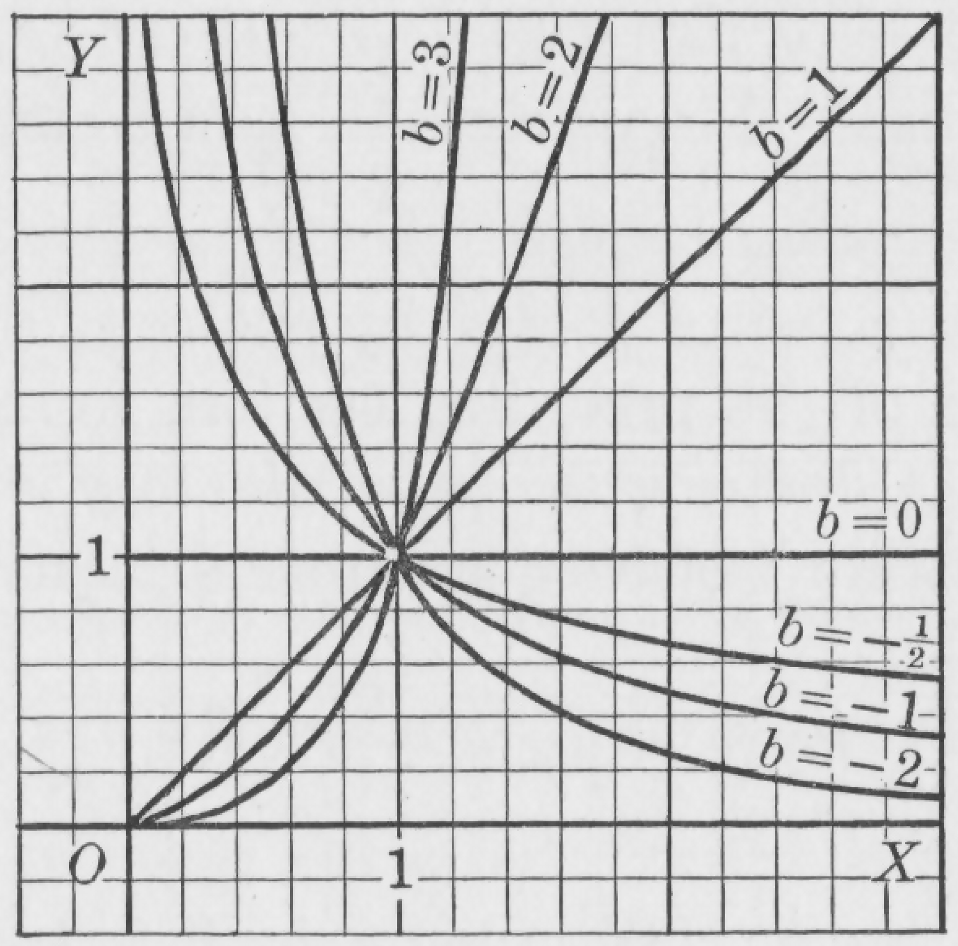


FIG. 126

A simple device for testing the adequacy of the power function for fitting a set of points is that of plotting the points on *logarithmic coördinate paper* which will be described in the next section. If the points thus plotted lie nearly in a straight line the power function may be used; otherwise not.

125. Logarithmic coördinates. Let us first draw an X-axis with abscissas measured in the usual manner; let X be a typical abscissa. To the point X attach now the number x, where

$$X = \log x.$$

The scale for x is called the **logarithmic scale**; that for X the **uniform scale.** In Figure 127 values of X are shown above

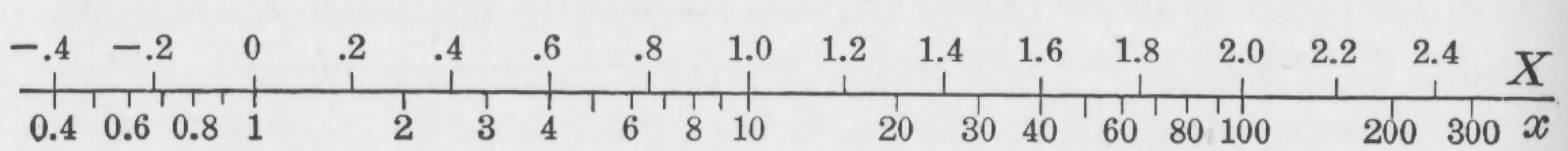


FIG. 127

the horizontal axis, and values of x below the axis. There are no negative values of x on a logarithmic scale.

If we replace both X and Y of a usual set of rectangular coördinates by logarithmic scales x and y we obtain a set of **logarithmic coördinates**. In Figure 128 is shown **logarithmic coördinate paper**. Sometimes decimal points and final zeros of numbers are not printed.

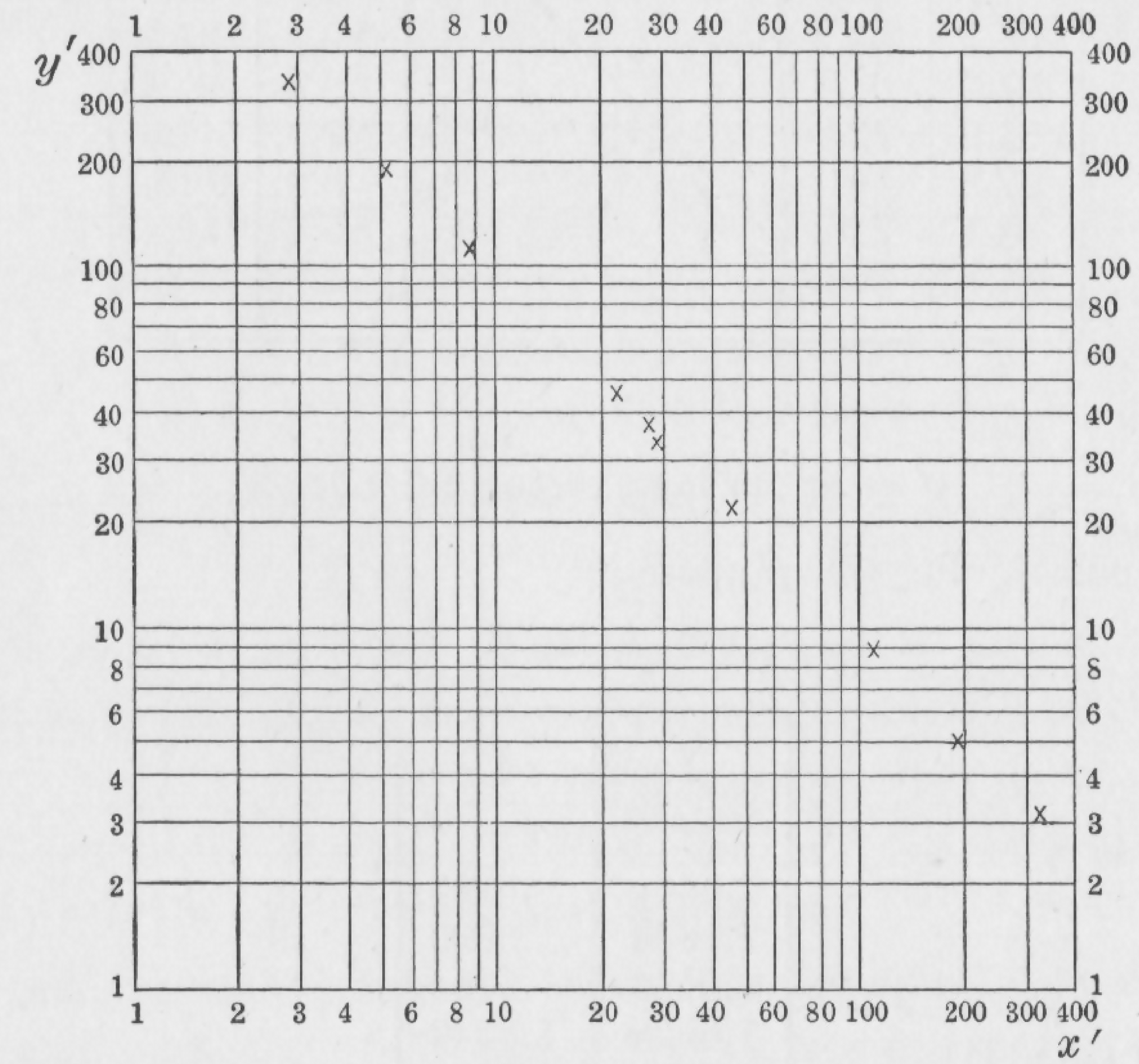


FIG. 128

Example. — In an experiment in an optical laboratory measurements were made as shown in Table I, p. 278. Find a formula connecting x and y.

Solution. — By plotting these points on ordinary rectangular coördinate paper we find that there appear to be horizontal and vertical asymptotes at about $y = 31$ and $x = 31$ respectively. If we set $x' = x - 31$, $y' = y - 31$, and plot the resulting points (x', y') on logarithmic coördinate paper, we find that the points deviate noticeably from a straight line. Trial shows, however, that if we set

$$x' = x - 29, \qquad y' = y - 28,$$

getting corresponding values of x' and y' as given in Table II, and plot these points on logarithmic coördinate paper, as shown in Figure 128, the points lie nearly in a straight line.

TABLE I

x	y
349	31.3
224	33.0
141	36.9
75	49.1
58.7	62.2
57.3	64.4
51.9	72.3
37.8	141.
34.1	223.
31.8	348.

TABLE II

x'	y'
320	3.3
195	5.0
112	8.9
46	21.1
29.7	34.2
28.3	36.4
22.9	44.3
8.8	113.
5.1	195.
2.8	320.

We therefore fit a curve of the type

$$y' = ax'^b \text{ or } \log y' = \log a + b \log x'$$

to these points. For this purpose set

$$X = \log x', \quad Y = \log y', \quad A = \log a.$$

TABLE III

X	Y
2.5051	.5185
2.2900	.6990
2.0492	.9494
1.6628	1.3243
1.4728	1.5340
1.4518	1.5611
1.3598	1.6464
.9445	2.0531
.7076	2.2900
.4472	2.5051

Corresponding values of X and Y are shown in Table III. If we fit a line

$$Y = A + bX$$

to these points by the method of average points we obtain

$$A = 3.0934, \quad b = -1.0384.$$

Since $A = \log a$, we have $a = 1240$. The required formula is therefore

$$y - 28 = 1240(x - 29)^{-1.0384}$$

or

$$\log (y - 28) = 3.0934 - 1.0384 \log (x - 29).$$

126. The exponential and logarithmic functions. A function of the form ab^x, where a and b are constants, will be called an **exponential function;** one of the form $a + b \log x$ a **logarithmic function.**

Let us see how a curve of the type

(1) $$y = ab^x$$

may be fitted to a set of points. Taking logarithms of both members we obtain

(2) $$\log y = \log a + x \log b.$$

Let

(3) $$Y = \log y, \quad A = \log a, \quad B = \log b;$$

then

(4) $$Y = A + Bx.$$

We replace the points (x_1, y_1), (x_2, y_2), $\cdots$ by the points (x_1, Y_1), (x_2, Y_2), $\cdots$, and fit a straight line to the latter set, thus finding A and B. Then we find a and b and substitute in (1) to get the desired equation.

To test whether an exponential function will give a satisfactory fit, we may plot the given points on **semi-logarithmic paper,** on which one scale is the uniform scale, the other the logarithmic scale, as shown in Figure 129. The use of the paper is similar to that of logarithmic paper.

By interchanging x and y in the preceding discussion we may fit a curve of the type

$$x = ab^y,$$

or, what is equivalent, the logarithmic type,

(5) $$y = a' + b' \log x.$$

Semi-logarithmic paper is again useful.

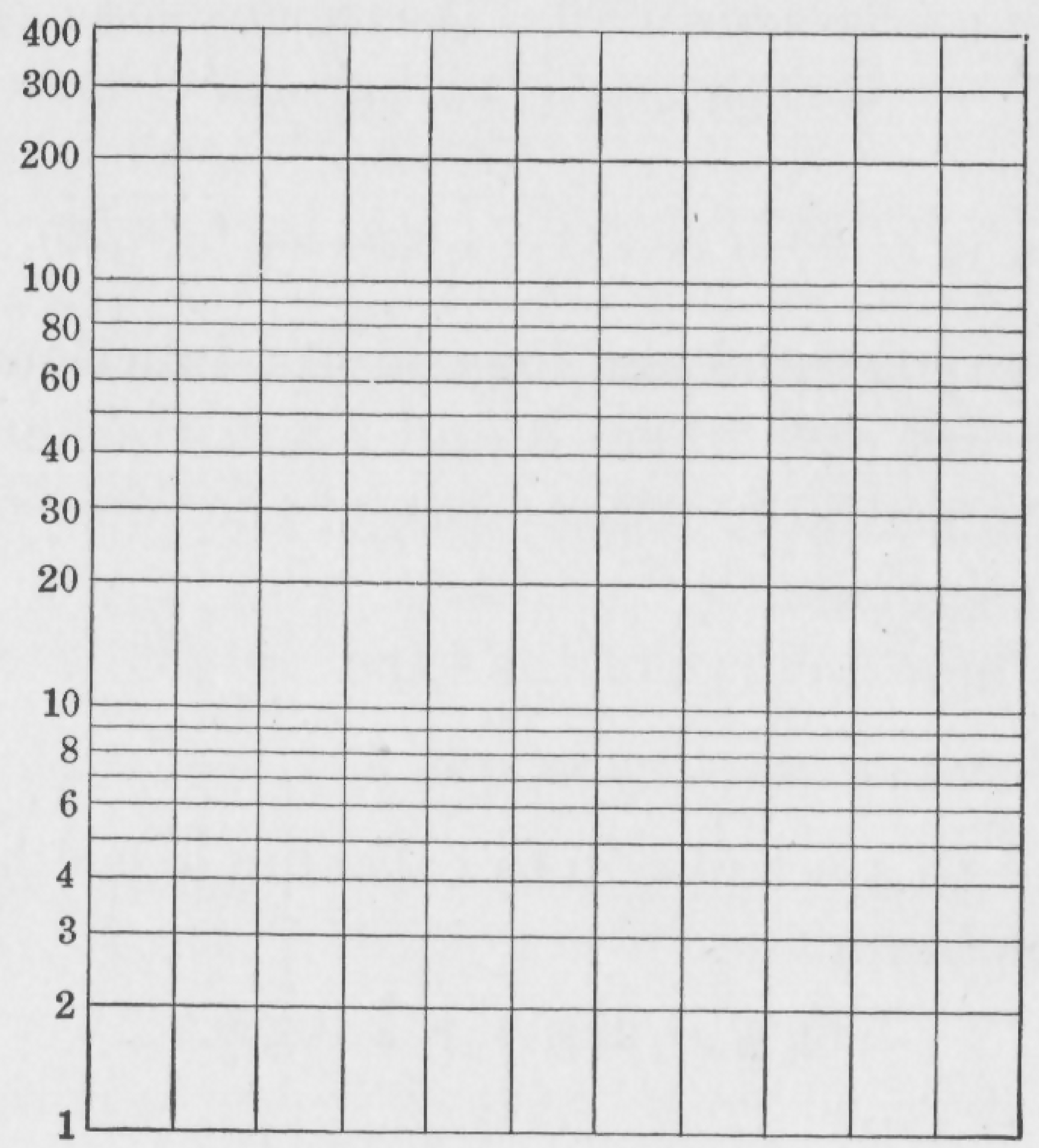


FIG. 129

EXERCISES

1. In a certain experiment a body fell a distance d in time t as shown in the following table:

t	1	2	3	4
d	7.51	28.55	62.77	116.4

(*a*) Plot the points (t, d) on rectangular coördinate paper. (*b*) Plot them on logarithmic coördinate paper. (*c*) Fit a curve of the type $d = at^b$ to the points and plot the curve on both the ordinary and the logarithmic coördinate paper.

2. In an experiment in physics the volume v of a certain amount of air depended upon the pressure p to which it was subjected, as shown in the following table:

p	100.8	98.23	95.82	91.32	89.10	86.81	84.33	78.20	69.60	55.92
v	26.45	27.05	27.70	29.12	29.88	30.62	31.48	34.18	37.98	47.10

(*a*) Plot the points on ordinary rectangular coördinate paper. (*b*) Plot them on logarithmic coördinate paper. (*c*) Fit a curve of the type $p = av^b$ to the points and plot the curve on both the ordinary and the logarithmic coördinate paper.

3. In testing a certain Mazda lamp in the physics laboratory, corresponding values of the voltage v and amperage i were observed to be as shown in the following table:

v	12	20	30	40	50	60	70	80	90	100
i	.06	.10	.17	.23	.29	.37	.43	.50	.57	.65

Find a formula of the type $i = av^b$ to fit the data. Represent the data and the formula graphically.

4. The population of the United States at each census from 1800 to 1870 is shown in the following table:

Year	1800	1810	1820	1830	1840	1850	1860	1870
Population in millions	5.31	7.24	9.64	12.87	17.07	23.19	31.44	38.56

Let P be the population in millions and t the number of the census starting with $t = 0$ in 1800. Fit a formula of the exponential type $P = ab^t$ to the data. Use the formula to find theoretical populations in 1880, 1900, 1920, and compare with the census reports which gave 50.16, 75.99, and 105.71 millions respectively. Draw a figure.

5. By use of semi-logarithmic paper find which of the equations $y = ab^x$ or $x = ab^y$ is best adapted to fit the data of the following table, and find a formula of that type which is satisfied approximately by the values given:

x	1	2	3	4	5	6	7	8
y	2.3	3.1	3.6	3.7	3.9	4.0	4.3	4.5

SOLID ANALYTIC GEOMETRY

CHAPTER XIV

PRELIMINARY DEFINITIONS AND FORMULAS

127. Rectangular coördinates. In plane analytic geometry the position of a point is determined by its directed distances from two mutually perpendicular lines, the axes of coördinates. In three dimensions the directed distances of a point from three mutually perpendicular planes serve to determine its position completely. These directed distances are the **rectangular coördinates** of the point with respect to the three planes of reference.

Let XOY, YOZ, and ZOX be the three mutually perpendicular planes of reference. They intersect by pairs in three mutually perpendicular lines OX, OY, OZ. These planes and lines are the **coördinate planes** and **axes** respectively; the point O is the **origin** of coördinates.

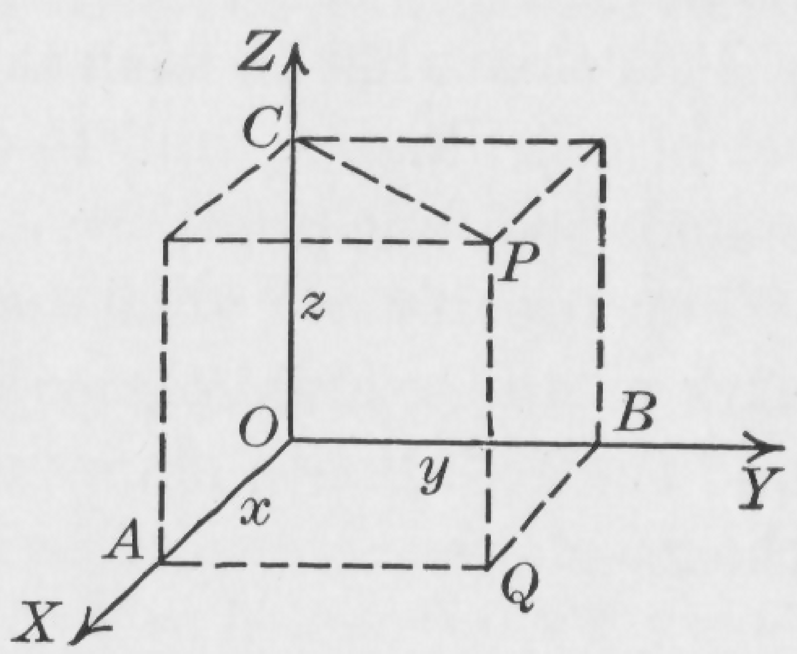


Fig. 130

An algebraic number scale is attached to each axis, as is done in plane analytic geometry, the point O being the zero point of each scale. The positive direction on each axis is indicated by the arrows in Figure 130. The units on the three axes are taken of equal length.

We now proceed to define the rectangular coördinates of a point P in terms of directed lengths on the coördinate axes or on parallels to them.

It is clear that the distance from the YOZ-plane to P may be determined by drawing through P a plane perpendicular to the axis OX; if this plane is cut by OX in the point A (the projection of P on OX), the distance from the YOZ-plane to P is OA. Let x be the number of the algebraic scale on OX corresponding to A. Then x is called the **x-coördinate** of P. Similarly the **y-coördinate** of P is the directed length OB cut off by a plane through P perpendicular to OY, and the **z-coördinate** is the directed length OC cut off by a plane through P perpendicular to OC. Thus we have

$$x = OA, \quad y = OB, \quad z = OC.$$

We designate P as the point $P(x, y, z)$.

If Q is the foot of the perpendicular from P to the XOY-plane, that is, the projection of P on that plane, it is evident that we may write

$$y = AQ, \quad z = QP,$$

provided AQ and QP are regarded as directed segments whose positive senses are the same as those of OY and OZ respectively.

It is clear that to each point of space corresponds just one set of coördinates; and to each set of real coördinates corresponds just one point.

For brevity we shall hereafter designate the coördinate axes as the **x-axis,** the **y-axis,** and the **z-axis** respectively, and the coördinate planes as the **xy-plane**, the **yz-plane,** and the **zx-plane.**

128. Drawings for figures. The representation of a three-dimensional figure on a plane sheet of paper may be made in many ways. The method most generally adopted is shown in Figure 130. Though sometimes called a "parallel projection" it is not a true projection in the proper sense of the word; it is an arbitrary conventional representation. The xy-plane is here considered as horizontal, the other two coördinate planes

as vertical, the yz-plane being regarded as coinciding with the sheet on which the figure is drawn.

The x-axis is represented by a line such that the angle XOY is about 135°. The unit on the x-axis is taken as about seven-tenths of the unit on OY and OZ. Lengths of two segments on the same line or on parallel lines are represented by proportional segments in the figure. Lines which are parallel are represented by parallel lines.

The position of a point $P(x, y, z)$ is indicated by drawing the dotted broken line $OAQP$ as shown in Figure 130.

Be able to ans.

EXERCISES

Draw a figure according to the directions of § 128 for each of Exercises 1 – 4 and plot the points indicated.

1. (3, 1, 4), (4, 3, 1), (4, − 1, 2).
2. (5, − 2, 4), (− 2, 5, 4), (− 3, 1, 2).
3. (4, 2, − 1), (3, 3, − 3), (3, 0, 3).
4. (3, − 3, 3), (− 1, 1, − 1), (0, − 2, 2).
5. Draw a figure according to the directions of § 128 in which P is so situated that all its coördinates are positive. By actual measurement determine the coördinates of P, taking the unit on OY as one-half of a centimeter (or one-fourth of an inch).
6. Proceed as directed in Exercise 5, except that P is to be so located that its x-coördinate is negative and its y- and z-coördinates are positive.
7. What equation is satisfied by all points of the xy-plane? Of the yz-plane? Of the zx-plane?
8. What is the locus of points for which both $y = 0$ and $z = 0$? For which both $x = 0$ and $y = 0$? For which both $x = 0$ and $z = 0$? For which $x = y = z$?
9. What are the coördinates of the point symmetric to $P(a, b, c)$ with respect to the origin? What are the coördinates of the points symmetric to $P(a, b, c)$ with respect to each coördinate plane? (P and P' are symmetric with respect to a plane if the segment PP' is perpendicular to the plane and is bisected by it.)
10. What are the coördinates of the points symmetric to $P(a, b, c)$ with respect to each of the coördinate axes?

study

129. Distance between two points. Let $P_1(x_1, y_1, z_1)$ and $P_2(x_2, y_2, z_2)$ be any two distinct points. Through each of them draw planes parallel to each of the coördinate planes. A rectangular box is thus formed for which P_1P_2 is a diagonal. If P_1A, P_1B, P_1C are edges of this box parallel respectively to the x-, y-, and z-axes, and if the fourth vertex of the base is designated D (see Fig. 131), it is easily seen that we have

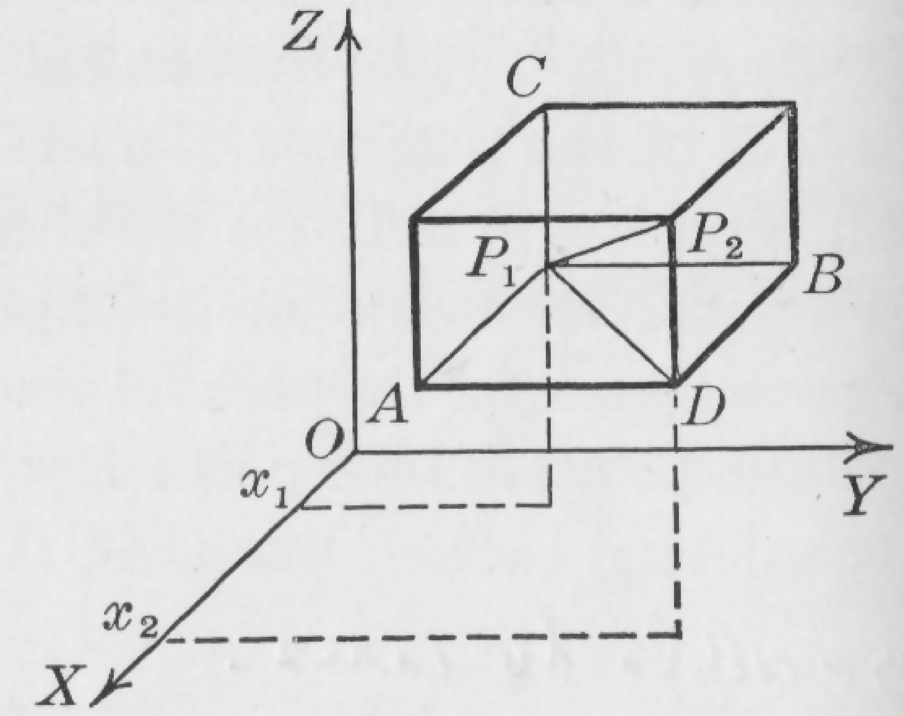


FIG. 131

$$(1)\quad P_1A = BD = x_2 - x_1, \quad P_1B = y_2 - y_1,$$
$$P_1C = DP_2 = z_2 - z_1;$$

$$(2)\quad \overline{P_1P_2}^2 = \overline{P_1D}^2 + \overline{DP_2}^2 = \overline{P_1D}^2 + (z_2 - z_1)^2;$$

$$(3)\quad \overline{P_1D}^2 = \overline{BD}^2 + \overline{P_1B}^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2.$$

We at once deduce from (2) and (3) the formula*

$$(4)\quad \overline{P_1P_2} = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}.$$

If P_2 is at the origin, (4) becomes

$$(5)\quad \overline{OP_1} = \sqrt{x_1^2 + y_1^2 + z_1^2}.$$

130. Equation of a sphere. Let $P(x, y, z)$ be any point on a sphere whose center is $Q(a, b, c)$ and whose radius is R. Since the length of QP is R, the equation

$$(1)\quad (x - a)^2 + (y - b)^2 + (z - c)^2 = R^2$$

holds for every point P on the sphere, and for no other points; it is an equation of the sphere. If the center is at the origin, (1) reduces to

$$(2)\quad x^2 + y^2 + z^2 = R^2.$$

* The box of Figure 131 flattens to a two dimensional figure if P_1 and P_2 are in a plane parallel to a coördinate plane, and becomes a line segment if P_1P_2 is parallel to a coördinate axis, but (4) remains true in all these cases.

When the operations indicated in (1) have been performed and the terms have been rearranged, the resulting equation has the form

$$(3)\quad x^2 + y^2 + z^2 + Ax + By + Cz + D = 0$$

where A, B, C, D are constants. Thus every sphere has an equation of form (3). The converse is not always true, but if (3) is written

$$\left(x + \frac{A}{2}\right)^2 + \left(y + \frac{B}{2}\right)^2 + \left(z + \frac{C}{2}\right)^2 = \frac{A^2 + B^2 + C^2 - 4D}{4}$$

we see that every equation (3) for which

$$A^2 + B^2 + C^2 - 4D > 0$$

is an equation of a sphere whose center is at the point $(-A/2, -B/2, -C/2)$, and whose radius R is given by

$$(4)\quad R = \tfrac{1}{2}\sqrt{A^2 + B^2 + C^2 - 4D}.$$

Example 1. — Find the locus of the equation

$$x^2 + y^2 + z^2 - 2x + 4y - 8z = 4.$$

Solution. — This may be written

$$(x - 1)^2 + (y + 2)^2 + (z - 4)^2 = 25.$$

The locus is therefore the sphere whose center is $(1, -2, 4)$ and whose radius is 5.

Example 2. — Find the equation of the sphere which passes through the points $(1, 1, 1)$, $(2, 2, 3)$, $(4, -2, 1)$, $(0, 0, 0)$.

Solution. — Substitute the coördinates of the given points in equation (3); we thus have

$$\begin{aligned} 3 + A + B + C + D &= 0,\\ 17 + 2A + 2B + 3C + D &= 0,\\ 21 + 4A - 2B + C + D &= 0,\\ D &= 0. \end{aligned}$$

The solution of these equations is $A = 1$, $B = 7$, $C = -11$, $D = 0$. Hence the desired equation is

$$x^2 + y^2 + z^2 + x + 7y - 11z = 0.$$

Skip

We leave it for the reader to verify that the equation to be found can be written in determinant form

$$\begin{vmatrix} x^2+y^2+z^2 & x & y & z & 1 \\ 1^2+1^2+1^2 & 1 & 1 & 1 & 1 \\ 2^2+2^2+3^2 & 2 & 2 & 3 & 1 \\ 4^2+(-2)^2+1^2 & 4 & -2 & 1 & 1 \\ 0^2+0^2+0^2 & 0 & 0 & 0 & 1 \end{vmatrix} = 0.$$

EXERCISES

1. Find the distances between the following pairs of points:

(*a*) (3, 3, 8) and (0, 0, 0);
(*b*) (4, 1, 2) and (4, 0, 2);
(*c*) (3, 1, 2) and (− 3, − 1, − 2);
(*d*) (4, 2, 3) and (4, − 2, − 3).

2. Find the distances between the following pairs of points:

(*a*) (0, 0, 0) and (− 1, − 2, 2);
(*b*) (2, − 2, 0) and (2, 1, 0);
(*c*) (2, 6, 9) and (− 2, − 6, − 9);
(*d*) (3, 2, 1) and (3, − 2, − 1).

3. Find an equation satisfied by the coördinates of every point $P(x, y, z)$ which is equidistant from two given points $P_1(x_1, y_1, z_1)$ and $P_2(x_2, y_2, z_2)$. Prove from this that every plane has an equation of the first degree in x, y, z.

4. Find an equation of the plane which perpendicularly bisects the line segment whose ends are the points (1, − 1, 2), (3, 0, 1).

5. Find an equation of the sphere whose center is the point (2, 4, 6) and which passes through the point (4, 0, 2).

6. Find the coördinates of a point equidistant from the four points (0, 2, 0), (0, 0, 4), (− 2, 0, 0), (4, 6, 4). Then find an equation of the sphere through the four points.

7. Find the center and radius of each of the spheres whose equations are

(*a*) $x^2 + y^2 + z^2 - 4x - 6y + 8z = 7$;
(*b*) $x^2 + y^2 + z^2 + 6x - 8y = 0$.

8. Find the center and radius of each of the spheres whose equations are

(*a*) $x^2 + y^2 + z^2 - 2y + 4z = 20$;
(*b*) $x^2 + y^2 + z^2 + 4x + 2y - 6z = 2$.

9. Find an equation of the sphere through the points: (2, 4, 6), (0, 2, 0), (0, 0, − 4), (0, 4, 2).

10. Find an equation of the sphere through the points: (0, 0, 0), (2, 2, 2), (− 2, 2, 0), (0, − 2, − 2).

131. Direction cosines of a line. Through any point P_1 of a directed line L draw P_1X', P_1Y', P_1Z' parallel respectively to the x-, y-, and z-axes, and similarly directed (see Fig. 132). The angles α, β, γ which L makes with P_1X', P_1Y', P_1Z' respectively are called the **direction angles** of L. These angles are to be taken positive or zero, and not greater than 180°.

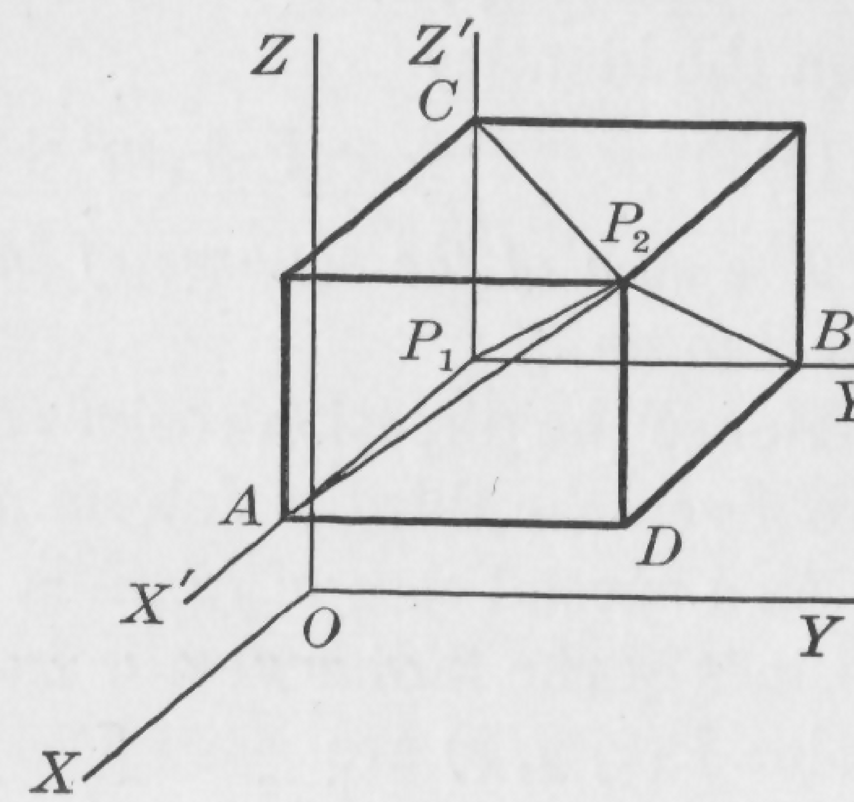


Fig. 132

The **direction cosines** of the line L are $\cos\alpha$, $\cos\beta$, $\cos\gamma$, this being the order in which we shall always hereafter write them. For brevity we shall designate them by l, m, n, respectively, so that

$$l = \cos\alpha, \quad m = \cos\beta, \quad n = \cos\gamma.$$

Let $P_1(x_1, y_1, z_1)$ and $P_2(x_2, y_2, z_2)$ be two points on the line L whose positive direction is from P_1 toward P_2. By drawing planes through P_1 and P_2 parallel to the coördinate planes, let us form the rectangular box of Figure 132. Since P_1A is perpendicular to the plane ADP_2, it is perpendicular to the line AP_2 in that plane. It follows that P_1AP_2 is a right triangle of reference for the angle α; hence $\cos\alpha = P_1A/\overline{P_1P_2}$. Similar considerations lead to corresponding expressions for $\cos\beta$ and $\cos\gamma$. Thus we have the three equations

$$(1) \qquad \cos\alpha = \frac{P_1A}{\overline{P_1P_2}}, \qquad \cos\beta = \frac{P_1B}{\overline{P_1P_2}}, \qquad \cos\gamma = \frac{P_1C}{\overline{P_1P_2}}.$$

From equations (1) of § 129 we have

$$P_1A = x_2 - x_1, \quad P_1B = y_2 - y_1, \quad P_1C = z_2 - z_1;$$

and let us write, in accordance with formula (4), § 129,

$$d_{12} = \overline{P_1P_2} = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}.$$

By combining these formulas we obtain the following set:

$$(2)\quad l = \frac{x_2 - x_1}{d_{12}}, \qquad m = \frac{y_2 - y_1}{d_{12}}, \qquad n = \frac{z_2 - z_1}{d_{12}}.$$

By squaring and adding these last three equations we obtain the identity

$$(3)\qquad \boldsymbol{l^2 + m^2 + n^2 = 1.}$$

The sum of the squares of the direction cosines of a line is equal to unity.

Hence the direction cosines are not independent; when two are given the third is determined, except for sign.

As a special case of formula (2) we note that the direction cosines of the radius vector from the origin $O(0, 0, 0)$ to the point $P(x, y, z)$ are

$$(4)\qquad l = \frac{x}{r}, \qquad m = \frac{y}{r}, \qquad n = \frac{z}{r},$$

where

$$r^2 = OP^2 = x^2 + y^2 + z^2.$$

In particular, the direction cosines of the positive x-axis are

$$l = 1, \qquad m = 0, \qquad n = 0.$$

Similarly, the direction cosines of the positive y-axis are

$$l = 0, \qquad m = 1, \qquad n = 0,$$

and those of the positive z-axis are

$$l = 0, \qquad m = 0, \qquad n = 1.$$

For configurations in the xy-plane, formulas for three dimensions should reduce to those of plane analytical geometry or trigonometry. This is often a useful check. For example, we observe that if a line lies in the xy-plane we have $\gamma = 90°$, and $\beta = \pm\ (90° - \alpha)$ or $90° + \alpha$ or $270° - \alpha$; hence

$$\cos\gamma = 0, \qquad \cos\beta = \pm \sin\alpha,$$

and (3) reduces to the familiar relation

$$\cos^2\alpha + \sin^2\alpha = 1.$$

132. Direction ratios. From equations (2) of § 131 it follows that

$$(1)\qquad l : m : n = (x_2 - x_1) : (y_2 - y_1) : (z_2 - z_1).$$

We may state the above formula as follows:

The direction cosines of a line are proportional to the projections of any segment of that line upon the coördinate axes.

This proposition is true no matter which direction on the line is taken as positive.

The proportion (1) is equivalent to two equations of the type

$$(2)\qquad \frac{l}{m} = \frac{x_2 - x_1}{y_2 - y_1}.$$

It is to be noted from equations (2) of § 131 that l, m, or n vanishes when, and only when, $x_2 - x_1$, $y_2 - y_1$, or $z_2 - z_1$, respectively, is zero. Hence at least one direction cosine is not zero. We take the two equations of type (2) so that the denominators are not zero.

If a, b, c are any three numbers such that

$$(3)\qquad l : m : n = a : b : c,$$

the ratios $a : b : c$ are called the **direction ratios** of the line whose direction cosines are l, m, n. If a, b, c are given, it is possible to determine l, m, n as follows: Equations (3) are equivalent to

$$(4)\qquad a = rl, \qquad b = rm, \qquad c = rn,$$

where r is some constant. If we square and add equations (4) we have, since $l^2 + m^2 + n^2 = 1$,

$$a^2 + b^2 + c^2 = r^2(l^2 + m^2 + n^2) = r^2.$$

Hence

$$r = \pm \sqrt{a^2 + b^2 + c^2},$$

and equations (4) become

$$(5)\quad \boldsymbol{l = \frac{a}{\pm\sqrt{a^2 + b^2 + c^2}}, \qquad m = \frac{b}{\pm\sqrt{a^2 + b^2 + c^2}},}$$

$$\boldsymbol{n = \frac{c}{\pm\sqrt{a^2 + b^2 + c^2}},}$$

the same sign being used with the radical in the three equations. It may be noted that to change the sign preceding the radical changes the angles α, β, γ to their supplements; this amounts only to a change in the direction that has been taken as positive on the line.

Example. — The direction ratios of a line are $1:-2:4$; find its direction cosines.

Solution. — By formula (5) we have

$$l = \frac{1}{\pm\sqrt{1^2 + (-2)^2 + 4^2}} = \frac{1}{\pm\sqrt{21}}; \quad m = \frac{-2}{\pm\sqrt{21}}; \quad n = \frac{4}{\pm\sqrt{21}}.$$

EXERCISES

In the following Exercises 1–4, find the direction cosines of each of the lines through the specified pairs of points.

1. (a) $P_1(2, 6, 0)$, $P_2(0, 2, 4)$; (b) $P_1(2, -4, -4)$, $P_2(2, 4, 2)$.

2. (a) $P_1(-1, 0, 0)$, $P_2(1, -1, 2)$; (b) $P_1(2, 2, -1)$, $P_2(-1, 5, 7)$.

3. (a) $P_1(0, 0, 0)$, $P_2(1, 1, -2)$; (b) $P_1(0, -1, 2)$, $P_2(1, 0, -2)$.

4. (a) $P_1(0, 0, 0)$, $P_2(1, -1, 1)$; (b) $P_1(2, 0, -2)$, $P_2(-2, 4, 2)$.

In the following Exercises 5–8, find the direction cosines of each of the lines having the given direction ratios.

5. (a) $1:2:-2$; (b) $3:0:-4$.

6. (a) $5:-12:0$; (b) $-1:4:8$.

7. (a) $1:0:1$; (b) $1:1:1$.

8. (a) $-1:1:1$; (b) $2:0:0$.

In the following Exercises 9–13 find the direction cosines, and the direction angles not specified, when the given relations exist among the direction angles.

9. (a) $\alpha = 60°$, $\beta = 45°$; (b) $\alpha = 135°$, $\beta = 90°$.

10. (a) $\alpha = 30°$, $\beta = 90°$; (b) $\alpha = 120°$, $\beta = 60°$.

11. (a) $\beta = 45°$, $\gamma = 45°$; (b) $\alpha = \beta = \gamma$.

12. (a) $\beta = 60°$, $\gamma = 150°$; (b) $\alpha = 180° - \beta = \gamma$.

13. (a) $\alpha = 45°$, $\beta = 135°$; (b) $\alpha = \beta = \frac{\gamma}{2}$.

14. Find the direction cosines of the line through the origin which makes equal angles with the positive x-, y-, and z-axes and which is directed into the first octant (the region where x, y, and z are all positive).

15. Find the direction cosines of the line in the zx-plane which bisects the angle between the positive x- and z-axes.

16. Find the direction cosines of the line in the xy-plane which passes through the origin and whose slope is $1/\sqrt{5}$.

17. Prove that if l, m, n are any three real numbers such that $l^2 + m^2 + n^2 = 1$, then there is a line whose direction cosines are l, m, n.

133. Parallel lines. If two line L_1 and L_2 are parallel, the corresponding direction angles are either all equal or all supplementary according as the positive senses on the lines are the same or not. Hence if the direction cosines of L_1 are l_1, m_1, n_1, and those of L_2 are l_2, m_2, n_2, we have either

$$l_1 = l_2, \quad m_1 = m_2, \quad n_1 = n_2,$$

or

$$l_1 = -l_2, \quad m_1 = -m_2, \quad n_1 = -n_2.$$

In either case,

$$l_1 : m_1 : n_1 = l_2 : m_2 : n_2. \tag{1}$$

That is, *if two lines are parallel their direction ratios are equal.*

Conversely, *if the direction ratios of two lines are equal, the lines are parallel.* To prove this, note that (1) is of the form of equation (3) of § 132 if we take l, m, n as l_1, m_1, n_1 and a, b, c as l_2, m_2, n_2; hence we may make these substitutions in equations (5) of § 132. If we recall that $l_2^2 + m_2^2 + n_2^2 = 1$, we see that the resulting equations may be written

$$l_1 = \pm l_2, \quad m_1 = \pm m_2, \quad n_1 = \pm n_2.$$

It follows that the direction angles of one line are either all equal or all supplementary to the corresponding direction angles of the other line, and the two lines must be parallel.

134. Angle between two lines. Perpendicular lines. Since two lines in space may not meet, we must define with some care the angle between them. We adopt the following definition:

By ***the angle θ between two directed lines,*** L_1 *and* L_2, *we mean the angle θ between directed rays L_1', L_2' whose initial points are at the origin, and which are parallel to, and have the same positive senses as,* L_1 *and* L_2 *respectively. We take θ as an angle not less than* $0°$ *nor more than* $180°$.

To obtain a formula for $\cos\theta$ in terms of l_1, m_1, n_1, the direction cosines of L_1 (and of L_1'), and of l_2, m_2, n_2, the direction cosines of L_2 (and of L_2'), we proceed as follows: Take P_1 and P_2 on L_1' and L_2' respectively so that

$$OP_1 = OP_2 = 1.$$

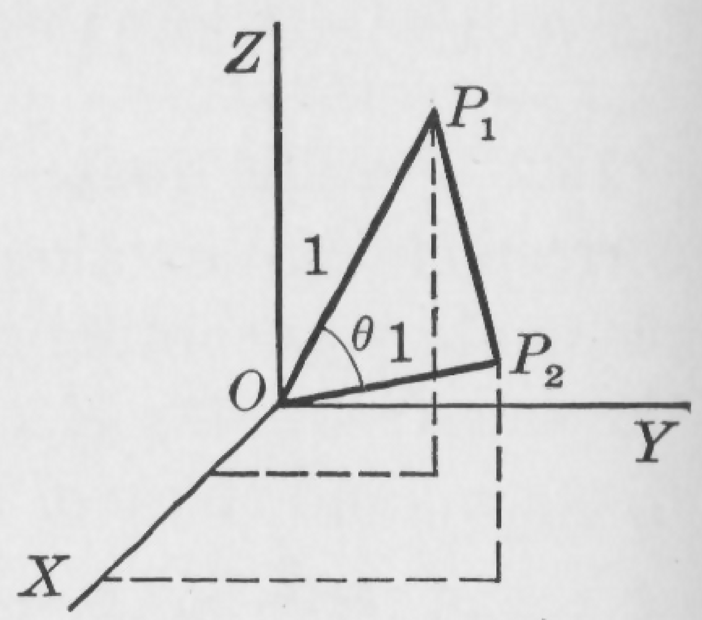


FIG. 133

Then, by formula (4) of § 131, the coördinates of P_1 are (l_1, m_1, n_1), and those of P_2 are (l_2, m_2, n_2). The Law of Cosines (see page 9) applied to the triangle OP_1P_2 gives

$$\text{(1)} \qquad \overline{P_1P_2}^2 = 1 + 1 - 2\cos\theta.$$

On the other hand we have, from the distance formula (4) of § 129 (page 286),

$$\begin{aligned}\text{(2)} \qquad \overline{P_1P_2}^2 &= (l_2 - l_1)^2 + (m_2 - m_1)^2 + (n_2 - n_1)^2 \\ &= (l_2^2 + m_2^2 + n_2^2) + (l_1^2 + m_1^2 + n_1^2) \\ &\qquad - 2(l_1l_2 + m_1m_2 + n_1n_2) \\ &= 1 + 1 - 2(l_1l_2 + m_1m_2 + n_1n_2).\end{aligned}$$

From (1) and (2) we derive the formula

$$\text{(3)} \qquad \cos\theta = \boldsymbol{l_1l_2 + m_1m_2 + n_1n_2}.$$

Since $\cos\theta = 0$ if and only if $\theta = 90°$, we have the following corollary of the proposition expressed by the preceding formula (3):

If the two lines L_1 *and* L_2 *are perpendicular, then*

$$\text{(4)} \qquad \boldsymbol{l_1l_2 + m_1m_2 + n_1n_2 = 0},$$

and conversely.

If $a : b : c$ are direction ratios of L_1, and $a_2 : b_2 : c_2$ are direction ratios of L_2, then from formula (5) of § 132 (page 291) and (3) of the present section we have

$$\text{(3')} \qquad \cos\theta = \frac{a_1a_2 + b_1b_2 + c_1c_2}{\pm\sqrt{a_1^2 + b_1^2 + c_1^2}\cdot\sqrt{a_2^2 + b_2^2 + c_2^2}}$$

and *the lines* L_1, L_2, *are perpendicular if and only if*

$$\text{(4')} \qquad a_1a_2 + b_1b_2 + c_1c_2 = 0.$$

Example. — Determine the direction ratios of a line L which is perpendicular to the line L_1 whose direction ratios are $1 : 2 : 0$, and which is also perpendicular to the line L_2 whose direction ratios are $0 : 2 : 1$.

Solution. — Let the direction ratios of L be $a : b : c$. The equations (4′) which express the fact that L is perpendicular to L_1 and L_2 respectively are

$$a + 2b = 0, \qquad 2b + c = 0.$$

Hence

$$\frac{a}{b} = \frac{-2}{1}, \qquad \frac{b}{c} = \frac{1}{-2},$$

so that

$$a : b : c = -2 : 1 : -2 = 2 : -1 : 2.$$

EXERCISES

1. Find all groups of parallel lines among the six lines through the following pairs of points:

(*a*) $P_1(2, 4, 0)$, $P_2(0, 2, 4)$; (*b*) $P_1(0, 0, 0)$, $P_2(1, 1, 1)$;
(*c*) $P_1(2, 1, 4)$, $P_2(1, 1, 5)$; (*d*) $P_1(1, -1, 2)$, $P_2(0, -2, 4)$;
(*e*) $P_1(0, 0, 1)$, $P_2(1, 1, -1)$; (*f*) $P_1(-1, 0, -1)$, $P_2(1, 2, 1)$.

2. Find all groups of parallel lines among the six lines whose direction ratios are as follows:

(*a*) $1 : 1 : -1$; (*b*) $1 : 0 : 1$; (*c*) $-2 : -2 : 2$;
(*d*) $-1 : -1 : 1$; (*e*) $-3 : 0 : -3$; (*f*) $1 : -2 : -1$.

3. Find all mutually perpendicular pairs of lines among the six described in Exercise 2.

4. Find all mutually perpendicular pairs of lines among the six described in Exercise 1.

Find the angle between each of the pairs of lines determined by the data for the following Exercises on page 292.

5. Exercise 1, page 292. **6.** Exercise 2, page 292.
7. Exercise 3, page 292. **8.** Exercise 4, page 292.

9. Exercise 5, page 292. **10.** Exercise 6, page 292.

11. Exercise 7, page 292. **12.** Exercise 8, page 292.

13. Find the direction cosines of a line perpendicular to each of the lines (*a*) and (*b*) of Exercise 1 of the present set.

14. Find the direction cosines of a line perpendicular to each of the lines (*e*) and (*f*) of Exercise 2 of the present set.

15. Show that the points (0, 1, 3), (− 1, 0, 2), (− 1, 1, 4) are vertices of a right triangle. Find its area.

16. Show that the points (0, 0, 0), (1, 1, − 2), (− 1, 1, − 3) are vertices of a right triangle. Find its area.

17. Show that the three points (1, 0, − 1), (2, 1, 3), (0, − 1, −5) are in a straight line.

18. Find the direction ratios of a line perpendicular to the plane of the triangle of Exercise 16.

19. Prove that the direction ratios $a : b : c$ of a line perpendicular to each of two non-parallel lines having the direction ratios $a_1 : b_1 : c_1$ and $a_2 : b_2 : c_2$ are given by the formula

$$a : b : c = \begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix} : \begin{vmatrix} c_1 & a_1 \\ c_2 & a_2 \end{vmatrix} : \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}.$$

20. Prove that if the direction ratios of three lines are $a_1 : b_1 : c_1$, $a_2 : b_2 : c_2$, $a_3 : b_3 : c_3$, respectively, the three lines have a common perpendicular if and only if

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = 0.$$

21. Show how formula (4′) may be transformed into the formula $m_1m_2 = -1$ (page 40) when the perpendicular lines L_1, L_2, lie in the xy-plane.

22. Show how formula (4) may be transformed into (1), page 40, when L_1 and L_2 lie in the xy-plane.

23. Find the coördinates of the mid-point of the line segment from $P_1(x_1, y_1, z_1)$ to $P_2(x_2, y_2, z_2)$.

24. Prove that if $P_0(x_0, y_0, z_0)$ divides the line segment from $P_1(x_1, y_1, z_1)$ to $P_2(x_2, y_2, z_2)$ in the ratio $r_1 : r_2$, then

$$x_0 = \frac{r_1x_2 + r_2x_1}{r_1 + r_2}, \qquad y_0 = \frac{r_1y_2 + r_2y_1}{r_1 + r_2}, \qquad z_0 = \frac{r_1z_2 + r_2z_1}{r_1 + r_2}.$$

CHAPTER XV

PLANES AND STRAIGHT LINES

135. Distance from a plane to a point.* We define the *normal line ON* to a given plane as the line through the origin perpendicular to the plane. The positive direction on ON is that from the origin toward the plane if the latter does not pass through the origin, but is chosen arbitrarily if the origin is a point of the plane.

Let p be the distance $\overline{OS}$ (positive or zero) on ON from O to the plane ABC, and let the direction cosines of the normal ON be l, m, n. Note that there is one and only one plane with given values of p (positive or zero), l, m, and n, provided the latter are direction cosines ($l^2 + m^2 + n^2 = 1$).

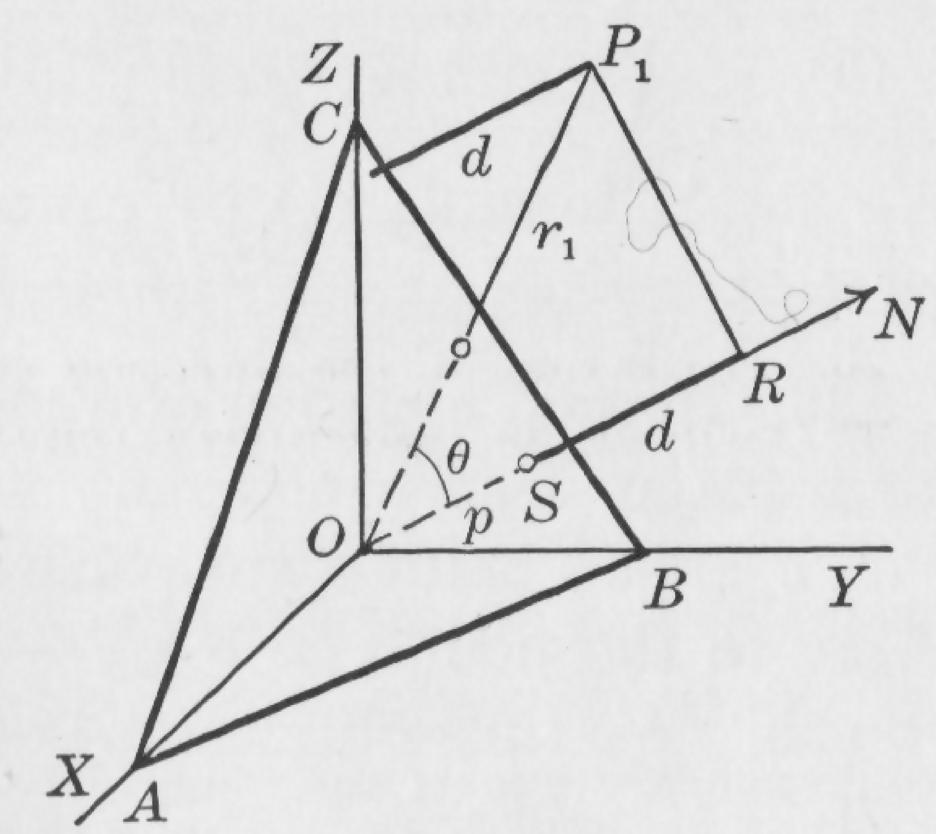


FIG. 134

The **directed distance *d* from the plane *ABC* to the point $P_1(x_1, y_1, z_1)$** is the measure of the directed segment SR on ON, where S is the point at which ON intersects the plane ABC and R is the foot of the perpendicular from P_1 to ON. The (undirected) distance from ABC to P_1 is equal to $\overline{SR} = |d|$, the numerical value of d. It is clear that d is zero if P_1 is on ABC, that d is positive if P_1 and O are on opposite sides of ABC, and that d is negative if P_1 and O are on the same side of ABC.

* Compare with § 30, page 73, *Distance from a line to a point.*

We shall now derive the following formula for d:

(1) $$\mathbf{d = lx_1 + my_1 + nz_1 - p.}$$

In the first place, we easily see that (1) is correct if P_1 is at the origin. If P_1 is elsewhere, take the positive direction on the line OP_1 as that from O to P_1, let $r_1 = \overline{OP_1}$, and let θ be the angle between OP_1 and ON. Then

(2) $$p + d = r_1 \cos \theta.$$

The direction cosines of OP_1 are

(3) $$l_1 = \frac{x_1}{r_1}, \quad m_1 = \frac{y_1}{r_1}, \quad n_1 = \frac{z_1}{r_1}.$$

By formula (3), page 294, we have

(4) $$\cos \theta = ll_1 + mm_1 + nn_1.$$

Hence from (2), (3), and (4) we have

(5) $$p + d = r_1\left(l\frac{x_1}{r_1} + m\frac{y_1}{r_1} + n\frac{z_1}{r_1}\right).$$

We obtain formula (1) by cancelling r_1 in (5) and solving for d.

136. Normal equation of a plane. If p, l, m, n are given for a plane, then a necessary and sufficient condition that $P(x, y, z)$ be a point of the plane is that the distance from the plane to the point, $lx + my + nz - p$ (formula (1), § 135) vanish. We thus obtain the **normal equation of a plane**

(1) $$\mathbf{lx + my + nz - p = 0,}$$

where p is the distance (positive or zero) from the plane to the origin, and l, m, n are cosine directors of the normal ON.

It follows that *every plane has a linear equation in x, y, z.* We shall see that the converse is also true, that is, that every linear equation has a plane as its locus. Before proving this, however, we shall first consider the problem of *reducing the general linear equation*

(2) $$Ax + By + Cz + D = 0$$

to normal form.

There is a line whose direction ratios are $A : B : C$; its direction cosines are, from formula (5) on page 291,

(3) $$l = \frac{A}{\pm\sqrt{A^2 + B^2 + C^2}}, \quad m = \frac{B}{\pm\sqrt{A^2 + B^2 + C^2}},$$
$$n = \frac{C}{\pm\sqrt{A^2 + B^2 + C^2}}.$$

This suggests dividing (2) by $\pm\sqrt{A^2 + B^2 + C^2}$. If we choose the $\pm$ sign so that

(4) $$p = -\frac{D}{\pm\sqrt{A^2 + B^2 + C^2}}$$

is positive when $D \neq 0$, we reduce (2) to the equivalent form

(5) $$lx + my + nz - p = 0.$$

Equation (5) is the equation of a plane in normal form; it is *the normal form for* (2). If $D = 0$, the $\pm$ sign to be taken in (3) is determined by the positive direction on ON.

Since (2) can always be reduced to (5), *every linear equation has a plane as its locus.*

It follows from the preceding discussion that *the direction ratios of the normal to a plane are $A : B : C$.*

The rule for finding the directed distance d from a plane to a line can now be stated in the following form:

Reduce the equation of the plane to normal form and substitute the coördinates of the point in the expression on the left side of this normal form; the resulting number gives the directed distance d *from the plane to the point.*

Example. — Find the distance from the plane
$$3x + 4y + 5 = 0$$
to the point (5, 0, 1).

Solution. — The equation is reduced to normal form by dividing by $-\sqrt{3^2 + 4^2 + 0^2} = -5$. The normal form is
$$-\tfrac{3}{5}x - \tfrac{4}{5}y - 1 = 0.$$
Hence
$$d = -\tfrac{3}{5} \cdot 5 - \tfrac{4}{5} \cdot 0 - 1 = -4.$$
The negative sign indicates that the origin and the point (5, 0, 1) are on the same side of the plane.

X all

EXERCISES

Write the following equations in normal form; find the distance of each plane from the origin and the direction cosines of its normal.

1. $x - 2y + 2z = 3.$ **2.** $x - 12y - 12z = 34.$

3. $3x - 4z + 5 = 0.$ **4.** $5y - 12z = 0.$

5. $8x + 15y = 0.$ **6.** $x + 4 = 0.$

In each of the following Exercises 7–12 find the distance from the plane whose equation is given to the point P *specified, and state on which side of the plane* P *lies.*

7. $2x + 4y - 4z = 9$, $P(-1, 1, 0)$.

8. $12x - 3y + 4z = 13$, $P(0, 0, 0)$.

9. $3x - 4y = 10$, $P(2, -2, 3)$.

10. $5y + 12z = 26$, $P(-1, -2, 0)$.

11. $x + 2y - 3z = 0$, $P(-1, -1, -1)$.

12. $2x - y = 0$, $P(1, 2, -1)$.

13. Find the equations of the planes which are at a distance of 4 units from the origin and perpendicular to the line joining the points $P_1(2, 3, 5)$, and $P_2(3, 1, 2)$.

14. Find the equations of the planes which are tangent to the sphere

$$x^2 + y^2 + z^2 = 36,$$

and perpendicular to the radii whose direction ratios are $2 : 3 : 4$.

15. Find the shortest distance from the plane

$$7x - 24y = 250,$$

tc the sphere

$$x^2 + y^2 + z^2 - 2x + 4y = 4.$$

16. Find the distance between the parallel planes

$$3x + 4y + 12z = 8,$$
$$6x + 8y + 24z = 3.$$

17. Find the equations of the planes which bisect the angles between the planes

$$3x + 4y + 12z - 13 = 0, \quad 7x - 24y + 50 = 0.$$

By showing that their normal lines are perpendicular, show that the two bisecting planes are mutually perpendicular.

18. Find the equations of the planes which bisect the angles between the planes

$$x - 2y + 2z = 6, \quad 3y - 4z = 0.$$

Show, as directed in Exercise 17, that the bisecting planes are mutually perpendicular.

✓ **137. Angles between two planes.** Let two planes intersect in a line AB. In the respective planes draw AL and AM perpendicular to AB. The **angles between the two planes** are LAM and its supplement, both being taken positive or zero and not greater than 180°. Let θ be the angle between directed normals to the planes. It is readily seen from Figure 135 that the angles between the two planes are θ and its supplement.

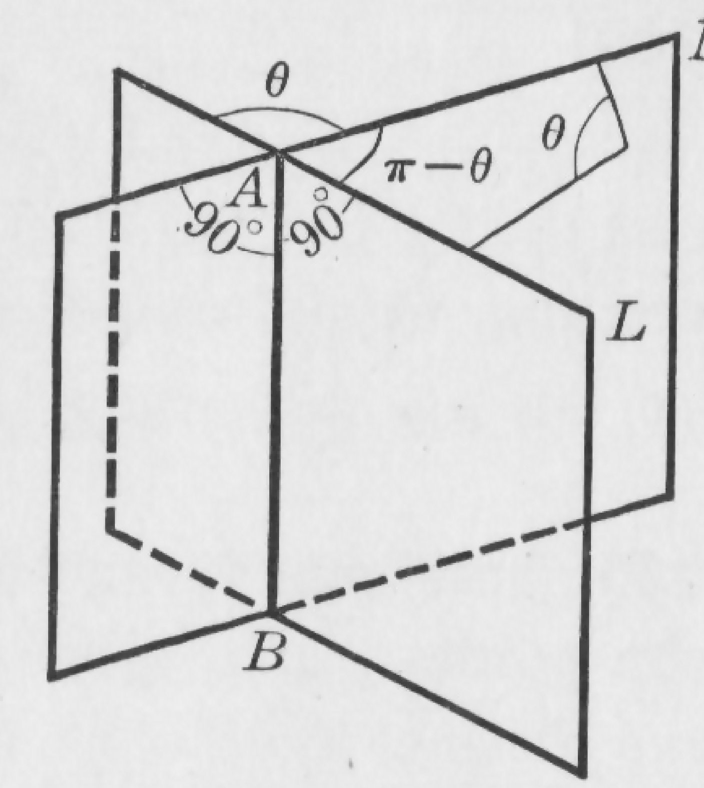


FIG. 135

If the equations of the planes are

$$(1) \qquad \begin{aligned} A_1x + B_1y + C_1z + D_1 &= 0, \\ A_2x + B_2y + C_2z + D_2 &= 0, \end{aligned}$$

the direction ratios for the normals to the planes are, by § 136, $A_1 : B_1 : C_1$ and $A_2 : B_2 : C_2$ respectively. Hence the angles between the two planes are the solutions for θ of the equations

$$(2) \qquad \cos\theta = \frac{A_1A_2 + B_1B_2 + C_1C_2}{\pm\sqrt{A_1^2 + B_1^2 + C_1^2} \cdot \sqrt{A_2^2 + B_2^2 + C_2^2}}.$$

The two planes are mutually perpendicular if and only if $\cos\theta = 0$, i.e.,

$$(3) \qquad A_1A_2 + B_1B_2 + C_1C_2 = 0.$$

The two planes (1) *are parallel or coincident if and only if their normals are parallel or coincident; hence, by* § 133, *if*

$$(4) \qquad A_1 : B_1 : C_1 = A_2 : B_2 : C_2.$$

138. Plane through a point and normal to a line. We seek an equation of the plane which passes through a given point $P_0(x_0, y_0, z_0)$ and is perpendicular to a line whose direction ratios are $a : b : c$. By § 136, the required equation is of the form

$$ax + by + cz + D = 0,$$

where D is a constant to be determined. Since P_0 lies on this plane its coördinates must satisfy the preceding equation, so that

$$ax_0 + by_0 + cz_0 + D = 0.$$

Substituting the value of D given by this equation in the preceding, we obtain the required equation:

$$(1) \qquad \mathbf{a(x - x_0) + b(y - y_0) + c(z - z_0) = 0.}$$

139. Intercept form. Equation of a plane through three given points. If a given plane cuts the coördinate axes in three distinct points, $(a, 0, 0)$, $(0, b, 0)$, and $(0, 0, c)$, then a, b, and c are called its intercepts.

The equation of the plane has the form

$$(1) \qquad Ax + By + Cz + D = 0.$$

Substitute the coördinate of the points in succession; we find that

$$Aa + D = 0, \qquad Bb + D = 0, \qquad Cc + D = 0.$$

Hence

$$A = -\frac{D}{a}, \qquad B = -\frac{D}{b}, \qquad C = -\frac{D}{c}.$$

When A, B, C have been given these values in (1), we transpose the last term to the other side of the equation and divide through by $-D$. The result is the **intercept form:**

$$(2) \qquad \frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1.$$

Let us now solve the more general problem of finding an equation of the plane that passes through any three given

points, $P_1(x_1, y_1, z_1)$, $P_2(x_2, y_2, z_2)$, $P_3(x_3, y_3, z_3)$, which do not lie on one line.

The equation has the form (1), where A, B, C are not all zero. If we substitute in succession the coördinates of the three points, we have

$$(3) \qquad \begin{aligned} Ax_1 + By_1 + Cz_1 + D &= 0, \\ Ax_2 + By_2 + Cz_2 + D &= 0, \\ Ax_3 + By_3 + Cz_3 + D &= 0. \end{aligned}$$

Since A, B, C are not all zero, it is possible to divide by one of these letters and obtain consistent equations in the ratios of the other three letters to this one. For example, if we divide by A we may have consistent equations in B/A, C/A, D/A. We solve for these ratios and substitute in (1) to obtain the desired equation.

The theory of determinants gives us a formula more readily written down; it is

$$(4) \qquad \begin{vmatrix} x & y & z & 1 \\ x_1 & y_1 & z_1 & 1 \\ x_2 & y_2 & z_2 & 1 \\ x_3 & y_3 & z_3 & 1 \end{vmatrix} = 0.$$

To prove that (4) is an equation of the required plane we need only note that it is of the first degree * in x, y, z, hence is the equation of a plane, and that the determinant in (4) vanishes when the coördinates of P_1, P_2, or P_3 are substituted for x, y, z.

Example. — Find the plane passing through the points $(3, 2, 1)$, $(5, 0, 2)$, $(0, -2, 3)$.

Solution. — The result of substituting these coördinates in (1) is the system of equations

$$\begin{aligned} 3A + 2B + C + D &= 0, \\ 5A \qquad + 2C + D &= 0, \\ -2B + 3C + D &= 0. \end{aligned}$$

* The coefficients of x, y, z in the expansion of (4) might all vanish, but this can happen only when P_1, P_2, P_3 are on one line.

By subtracting pairs of these equations we obtain a pair in A, B, C. These can be solved for A/C and B/C, and we can then substitute these values in one of the preceding equations and solve for D/C. The results are

$$\frac{A}{C} = 0, \quad \frac{B}{C} = \frac{1}{2}, \quad \frac{D}{C} = -2.$$

If we divide (1) by C and substitute the above values for the new coefficients we have, on simplifying, the required equation

$$y + 2z - 4 = 0.$$

The evaluation of the determinant form (4) is left as an exercise for the reader.

EXERCISES

Find the angles between the planes of each pair whose equations are given as follows. If the planes of a pair are parallel or perpendicular, indicate the fact.

1. $3x + 4y + 12z = 13, \quad 3x - 4z = 5.$
2. $x + y = 2, \quad y + z = 3.$
3. $x + 2y - z = 4, \quad x - y - z = 0.$
4. $2x - y + 2z = 3, \quad -4x + 2y - 4z = 9.$
5. $2x + 2y - 4z = 1, \quad -3x - 3y + 6z = 4.$
6. $x - z = 0, \quad y = 0.$
7. $3x + 3y - 8z = 5, \quad x - 2y + 2z = 2.$
8. $x + 4y - 8z = 0, \quad z = 0.$

Find an equation of each of the planes through the following points perpendicular to the lines indicated.

9. Point $(2, 3, 4)$; direction ratios of perpendicular, $3 : 4 : 12$.
10. Point $(0, -2, 0)$; direction ratios of perpendicular, $1 : 0 : -1$.
11. Point $(-1, 2, -1)$; perpendicular passes through $(3, 1, 1)$, $(2, 0, 3)$.
12. Point $(0, -1, 2)$; perpendicular passes through $(0, 0, 0)$, $(1, -4, 3)$.

Find an equation of each of the planes through the following groups of three points.

13. $(2, 0, 0), (0, -1, 0), (0, 0, 4)$.
14. $(0, -2, 0), (-1, 0, 0), (0, 0, -3)$.

15. $(0, 4, 4), (2, 2, 2), (3, 1, 4)$.
16. $(2, 1, 4), (0, 0, 0), (4, -3, 2)$.
17. $(0, 0, 1), (1, 0, 0), (0, 0, 0)$.
18. $(0, 1, -1), (-3, 2, 2), (2, -2, -4)$.

Solve the following Exercises.

19. Find an equation of a plane through the point $(1, -1, 1)$, and parallel to the xy-plane.

20. Find an equation of the plane through the point $(-1, 0, 1)$, and parallel to the plane $x - 2y + z = 0$.

21. Find an equation of the plane through the point $(0, -2, 2)$, and passing through the x-axis.

22. Find an equation of the plane through the origin, and perpendicular to each of the planes $x - y - z = 0$, $2x + 2y = 1$.

23. Find an equation of the plane whose x-intercept is a, whose y-intercept is b, and which is parallel to the z-axis.

24. Find the value of $\cos\theta$, where θ is an angle between the two planes $y = m_1x$, $y = m_2x$. Find $\tan\theta$ and compare with formula (1), page 40. Explain the similarity.

140. Equations of a straight line. Let $P_0(x_0, y_0, z_0)$ be a point on a straight line whose direction ratios are $a : b : c$. Let $P(x, y, z)$ be any other point on the line. Then, by § 132,

$$(x - x_0) : (y - y_0) : (z - z_0) = a : b : c. \tag{1}$$

This may be written

$$\frac{x - x_0}{a} = \frac{y - y_0}{b} = \frac{z - z_0}{c}. \tag{2}$$

These equations are satisfied by the coördinates of P if P lies on the line, and not otherwise. They are called **symmetric equations** of the line.

It should be noticed that there are two independent equations in (2), for example,

$$\frac{x - x_0}{a} = \frac{y - y_0}{b}, \qquad \frac{y - y_0}{b} = \frac{z - z_0}{c}. \tag{2'}$$

Even if a, b, or c is zero we use the formula (2); but if a, for example, is zero, we take the equivalent system (2′) as

$$x - x_0 = 0, \qquad \frac{y - y_0}{b} = \frac{z - z_0}{c},$$

and if $a = b = 0$ we take for (2′)

$$x - x_0 = 0, \qquad y - y_0 = 0.$$

Each of the equations (2′) is the equation of a plane; the locus of the simultaneous equations is the line of intersection of the two planes.

A straight line is represented by a pair of simultaneous linear equations.

Equations of any two planes intersecting in a given line are, taken simultaneously, equations of the line. A line may be represented by any one of the infinitely many pairs of such equations.

Equations (2′) are particularly simple in that each equation contains only two variables. The first of equations (2′) is the equation of a plane through the line and parallel to the z-axis. The locus of this equation in the plane analytic geometry of the xy-plane is the line obtained by projecting the given line on that plane. The plane

$$\frac{x - x_0}{a} = \frac{y - y_0}{b}$$

is therefore sometimes called the *plane of projection on the* xy-*plane*. Similar interpretations may be given for the other equations of (2).

Let $P_1(x_1, y_1, z_1)$ and $P_2(x_2, y_2, z_2)$ be two points of a line. The direction ratios of the line are, by § 132,

$$(x_2 - x_1) : (y_2 - y_1) : (z_2 - z_1).$$

Hence, from (2), equations of the line are

$$(3) \qquad \frac{x - x_1}{x_2 - x_1} = \frac{y - y_1}{y_2 - y_1} = \frac{z - z_1}{z_2 - z_1}.$$

This is the **two point form.**

If, when a, b, and c are given numbers, we let t be the value of each fraction in (2), then we may write

$$(4) \qquad x = x_0 + at, \qquad y = y_0 + bt, \qquad z = z_0 + ct.$$

For each point P the value of t is uniquely determined; but it varies with P. Conversely for each value of t there is a unique point $P(x, y, z)$, and P varies with t. By giving all values to t we get all points of the line and no others. These equations (4) are called **parametric equations of the line,** t being the parameter.

Example. — Find equations of the line which passes through the point (2, 1, 2) and is perpendicular to the plane $2x - 3y + 4 = 0$.

Solution. — By § 136, the direction ratios of the normal to the plane, and hence of the required line, are $2 : -3 : 0$. By (2), required equations are

$$(5) \qquad \frac{x - 2}{2} = \frac{y - 1}{-3} = \frac{z - 2}{0}.$$

In accordance with the remarks on (2) and (2′), equations (5) are equivalent to

$$\frac{x - 2}{2} = \frac{y - 1}{-3}, \quad \text{and} \quad z - 2 = 0.$$

The first of these equations is the same as the first of (5); the second would be obtained formally from the last of equations (5) by clearing of fractions.

Parametric equations of the line are

$$x = 2 + 2t, \quad y = 1 - 3t, \quad z = 2.$$

141. Pairs of linear equations. In the preceding section we have remarked that the locus of two simultaneous linear equations

$$(1) \qquad \begin{cases} A_1x + B_1y + C_1z + D_1 = 0, \\ A_2x + B_2y + C_2z + D_2 = 0, \end{cases}$$

is the line of intersection of the corresponding planes (provided they intersect). The equation

$$(2) \quad k_1(A_1x + B_1y + C_1z + D_1) + k_2(A_2x + B_2y + C_2z + D_2) = 0,$$

where k_1 and k_2 are constants not both zero, represents a plane through the line of intersection of the planes (1).* Any two different equations (2) taken simultaneously therefore have this same line as their locus.

In particular we may choose k_1 and k_2 so that one variable, say z, is eliminated. The resulting equation (2) represents a plane through the line and parallel to the z-axis, — it is the plane which projects the line on the xy-plane. Thus the equations of a line may be taken as those of two of the planes which project it on coördinate planes. The equations (2′) of § 140 are of this type, as are any two of equations (3) of § 140.

A problem of interest is the reduction of a pair of equations (1) to the standard form (2) of § 140. To solve this problem we first find the direction ratios $a : b : c$ of the line L represented by (1).

Since L is perpendicular to the normal to the first plane, of which the direction ratios are $A_1 : B_1 : C_1$ (§ 136) we have (§ 134)

$$A_1a + B_1b + C_1c = 0.$$

Likewise we must have

$$A_2a + B_2b + C_2c = 0.$$

We solve these two equations for the ratios $a : b : c$. The result may be written

$$(3)\quad a : b : c = (B_1C_2 - B_2C_1) : (C_1A_2 - C_2A_1) : (A_1B_2 - A_2B_1)$$

$$= \begin{vmatrix} B_1 & C_1 \\ B_2 & C_2 \end{vmatrix} : \begin{vmatrix} C_1 & A_1 \\ C_2 & A_2 \end{vmatrix} : \begin{vmatrix} A_1 & B_1 \\ A_2 & B_2 \end{vmatrix}.$$

Having found the ratios $a : b : c$ we now determine a point (x_0, y_0, z_0) whose coördinates satisfy (1).† We can at once write down the equations (2) of § 140.

* The coefficients of x, y, and z in (2) cannot all cancel out unless the planes (1) are parallel or coincident.

† This can be done by taking, for example, x_0 arbitrarily and substituting it for x in (1), then solving for $y = y_0$ and $z = z_0$.

To reduce (1) to form (3) of § 140, we have only to determine two points that satisfy (1) and substitute their coördinates in formula (3), § 140.

Example 1. — Reduce the equations of the line

$$(4)\quad \begin{cases} 2x + 3y - z = 0, \\ 3x - y + z = 2, \end{cases}$$

to standard forms.

Solution. — (*a*) Equations (3) give

$$a : b : c = (3 - 1) : (-3 - 2) : (-2 - 9) = 2 : -5 : -11.$$

To find a point on (4) substitute $x = x_0 = 0$ and solve for $y = y_0$, $z = z_0$. The results are

$$x_0 = 0, \quad y_0 = 1, \quad z_0 = 3.$$

With these values for a, b, c, x_0, y_0, z_0, formula (2) of § 140 becomes

$$(5)\quad \frac{x - 0}{2} = \frac{y - 1}{-5} = \frac{z - 3}{-11}.$$

The direction cosines of the line are

$$l = \frac{2}{\pm\sqrt{150}}, \quad m = \frac{-5}{\pm\sqrt{150}}, \quad n = \frac{-11}{\pm\sqrt{150}}.$$

(*b*) We have determined one point, (0, 1, 3), of the line. If we substitute $y = 0$ in equations (4) and solve for x and z, we obtain another point, (2/5, 0, 4/5), on the line. If we use these two points as (x_1, y_1, z_1), (x_2, y_2, z_2), formula (3) of § 140 becomes

$$(6)\quad \frac{x - 0}{\frac{2}{5} - 0} = \frac{y - 1}{0 - 1} = \frac{z - 3}{\frac{4}{5} - 3},$$

which could be somewhat simplified by multiplying each denominator by 5. The result is at once identified with (5).

(*c*) In parametric form, equation (4) reduces, through (5), to

$$(7)\quad x = 2t, \quad y = 1 - 5t, \quad z = 3 - 11t.$$

Example 2. — Find equations of the planes which project the line (4) on each of the coördinate planes.

Solution. — The required equations are obtained by choosing k_1 and k_2 in (2) so as to eliminate, in turn, x, y, and z. Thus, to eliminate z from equations (4) we take $k_1 = k_2 = 1$ and obtain

$$5x + 2y - 2 = 0,$$

which, interpreted in three dimensions, is the equation of the plane projecting the line (4) on the xy-plane, and in two dimensions is the equation of the projected line. The equations in three dimensions of this projected line are

$$5x + 2y - 2 = 0, \quad z = 0.$$

The other required equations are

$$11x + 2z - 6 = 0, \quad 11y - 5z + 4 = 0.$$

Example 3. — Find the equation of the plane passing through the point (3, 2, 4) and through the line

$$\begin{cases} 2x + y - z = 2, \\ x + 2y + 3z = 1. \end{cases}$$

Solution. — The equation

$$(8) \qquad k_1(2x + y - z - 2) + k_2(x + 2y + 3z - 1) = 0$$

represents a plane through the given line; it will also pass through (3, 2, 4) if we determine k_1 and k_2 so that (8) is satisfied by $x = 3$, $y = 2$, $z = 4$. These substitutions in (8) give the equation

$$2k_1 + 18k_2 = 0.$$

We can therefore take $k_1 = 9$, $k_2 = -1$. With these values for k_1 and k_2, (8) becomes

$$17x + 7y - 12z - 17 = 0.$$

EXERCISES

Find equations for each of the straight lines which satisfy the following conditions in Exercises 1–12.

1. The line passes through the point (3, 1, 2) and has the direction ratios $2 : 3 : -2$.

2. The line passes through the origin and has the direction ratios $3 : 0 : -1$.

3. The line passes through the point (0, 0, 4) and has the direction ratios $1 : 0 : 0$.

4. The line passes through the point (7, 2, 1) and makes equal angles with OX, OY, and OZ.

5. The line passes through the points (2, 1, 2) and (4, − 1, 0).

6. The line passes through the points (4, − 2, 0) and 2 − 2, 2).

7. The line passes through the points (2, 0, 4) and (4, 0, 4).

8. The line passes through the points (0, 1, − 2) and (0, − 1, 2).

9. The line passes through the origin and is perpendicular to the plane $x + y + 2z = 4$.

10. The line passes through the point (2, 1, 3) and is perpendicular to the plane $3x - 2y + 2z = 1$.

11. A segment of the line is the diameter through the point (0, 1, 2) of the sphere $x^2 + y^2 + z^2 + 2x + 4y = 9$.

12. The line passes through the point (1, − 1, 3), intersects the z-axis, and is perpendicular to the z-axis.

13. Find parametric equations of the lines in Exercises 1 and 2.

14. Find parametric equations of the lines in Exercises 5 and 6.

15. Find parametric equations of the line through $P_0(x_0, y_0, z_0)$ which has direction cosines l, m, n, using as a parameter the distance s from the point $P_0(x_0, y_0, z_0)$ to a variable point $P(x, y, z)$ on the line.

16. Find the direction cosines of the line

$$\begin{cases} 2x + y + 2z = 5, \\ 3x + y + z = 0, \end{cases}$$

and reduce these equations to the symmetric form.

17. Proceed as directed in Exercise 16 for the line

$$\begin{cases} 4x - y - z = 2, \\ 12x + y - z = 4. \end{cases}$$

18. Find equations of the planes which project the line of Exercise 16 on the coördinate planes.

19. Find the projections on the coördinate planes of the line of Exercise 17.

20. Find the equation of the plane which passes through the line of Exercise 16 and through the origin.

MISCELLANEOUS EXERCISES

1. Find the intercepts of the plane $x - y + 2z = 4$.

2. Find the point of intersection of the three planes

$$x + 2y - z + 3 = 0, \quad 2x - y + 2z = 11, \quad y - x + z = 0.$$

3. Find the points in which the line

$$\frac{x - 1}{2} = \frac{y - 1}{3} = \frac{z + 1}{-1}$$

meets the coördinate planes.

4. Find symmetric equations of the lines in which the plane $x - y + 2z = 4$ meets the coördinate planes.

5. On which side of the plane $2x - 3y + z = 5$ do the following points lie: $(1, 1, 2)$, $(3, 0, 0)$, $(0, -1, 2)$, $(2, 0, 2)$?

6. Find equations of the planes one unit distant from the origin which pass through the line

$$\begin{cases} x - y - z = 0 \\ y + z = 3. \end{cases}$$

7. Prove that the line of Exercise 6 lies in the plane $x + y + z = 6$.

8. Find an equation of the plane through the line of Exercise 6 and perpendicular to the plane $x - y + 2z = 2$.

9. Find an equation of the plane through the points $(2, -1, 2)$, $(1, 0, -1)$ and perpendicular to the plane $3x + y + 2z = 0$.

10. Find symmetric equations of the line which passes through the point $(-2, 0, 1)$ and is parallel to each of the planes

$$x - y = 0, \quad 2x + y - z = 0.$$

11. Find the cosine of an angle between the two lines

$$\begin{cases} 4x - y + z + 1 = 0, \\ 8x - y + 1 = 0, \end{cases} \qquad \begin{cases} 4x + y + z = 9, \\ y - z + 1 = 0. \end{cases}$$

12. Find symmetric equations of a line perpendicular to each of the lines of Exercise 11.

13. For what value of k do the three points $(k, 1, 2)$, $(2, -k, 3)$, $(3, -3, 4)$ lie on one straight line?

14. Find equations of the locus of a point equidistant from the three points $(0, 0, 1)$, $(0, -2, 0)$, $(-1, 0, 0)$.

15. The perpendicular from $(2, -1, 3)$ to a plane meets that plane in the point $(-1, 0, 1)$. Find an equation of the plane.

16. If a, b, c are the intercepts of a plane, and p is the distance from the plane to the origin, prove that

$$\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2} = \frac{1}{p^2}.$$

17. Find the perpendicular distance from the point $(1, -1, 1)$ to the line which joins the points $(4, 0, -2)$, $(-1, 0, 1)$.

Hint. Find the point in which the plane through the first given point perpendicular to the given line cuts that line.

18. Find equations of the line through the point $(1, -1, 1)$ that intersects perpendicularly the line which joins the points $(4, 0, -2)$, $(-1, 0, 1)$.

19. Find the area of the triangle whose vertices are $(1, -1, 1)$, $(4, 0, -2)$, $(-1, 0, 1)$.

20. Find the volume of the tetrahedron whose base, of area 10, lies in the plane $2x - y + z = 1$, and whose fourth vertex is $(2, 3, 2)$.

21. Prove that an angle θ between a line which has direction cosines l, m, n, and a plane whose normal has direction cosines l', m', n' is a solution of the equation

$$\sin \theta = ll' + mm' + nn'.$$

22. Prove that four points (x_1, y_1, z_1), (x_2, y_2, z_2), (x_3, y_3, z_3), (x_4, y_4, z_4) lie in a plane if and only if

$$\begin{vmatrix} x_1 & y_1 & z_1 & 1 \\ x_2 & y_2 & z_2 & 1 \\ x_3 & y_3 & z_3 & 1 \\ x_4 & y_4 & z_4 & 1 \end{vmatrix} = 0.$$

23. Write an equation in determinant form which is a necessary condition that three planes, no two of which are parallel, whose equations are given, have a common line of intersection.

24. Express in determinant form a condition that three lines through the origin with given direction cosines lie in one plane.

CHAPTER XVI

SURFACES AND CURVES

142. Equations and loci in three dimensions. We have seen that a plane is the locus of a single equation of the first degree in three variables interpreted as rectangular coördinates. By analogy with the corresponding situation in two dimensions we should expect that in general the locus in space of an equation other than one of the first degree would be a curved surface. This is, in fact, the case, as will be illustrated in the present chapter.

We shall also see that, if they have real solutions, two simultaneous equations in x, y, z, represent a curve in space, the intersection of the two corresponding surfaces.

143. Cylinders. It is natural to begin the discussion of equations in three dimensions by first considering those in which at least one of the variables x, y, z is missing. We shall see that the corresponding loci are cylinders.

Consider the locus in space of the equation

$$x^2 + y^2 = a^2. \tag{1}$$

Suppose $P_1(x_1, y_1, 0)$ is any point which satisfies the equation. Then P_1 lies on a circle in the xy-plane, with center at the origin and radius a. It is seen that $Q_1(x_1, y_1, z_1)$, where z_1 has any value whatever, also satisfies the equation. By giving all values to z_1 we get all points on the line which passes through P_1 and is parallel to the z-axis. Hence every point on the right circular cylinder whose axis is the z-axis and whose radius is a lies on the locus of (1). Also points not on this cylinder will not satisfy the equation. The locus of (1) is therefore precisely that cylinder.

Consider the locus in space of the equation

$$x^2 = 2py. \tag{2}$$

As before, let $P_1(x_1, y_1, 0)$ be a point on the locus; it must lie on a certain parabola in the xy-plane. Moreover the point $Q_1(x_1, y_1, z_1)$, where z_1 has any value whatever, also satisfies the equation, so that all lines through the points of the parabola and parallel to the z-axis lie in the locus; and no point not on one of these lines will satisfy (2). The locus is called a **parabolic cylinder** (Fig. 136).

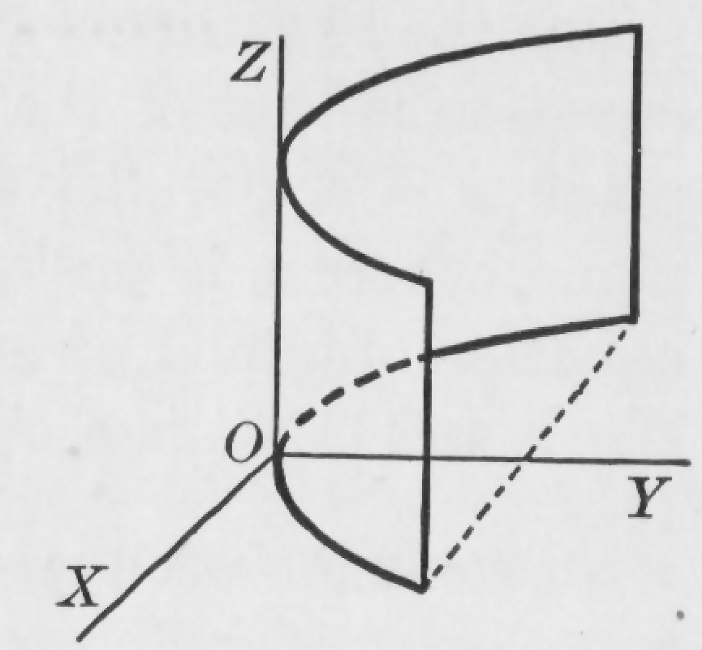


FIG. 136

In general the locus of an equation*

$$f(x, y) = 0,$$

in which z is missing, is a surface made up of straight lines parallel to the z-axis, passing through all points of the curve in the xy-plane whose equation in that plane is $f(x, y) = 0$. The surface is called a **cylinder** in every case. The plane curve is the **generatrix** and the lines are **elements** of the cylinder.

It is obvious that the loci of equations of the forms

$$f(x, z) = 0, \quad f(y, z) = 0,$$

are cylinders with elements parallel to the y- and x-axes respectively.

Note that planes parallel to the coördinate planes are special cases of cylinders, corresponding to equations in one variable only.

In the following sections we shall consider equations reducible to the type $f(x, y, z) = 0$. To each value of x, y one or more values of z (real or imaginary) correspond, so that the loci, if real, consist of one or more surfaces.

* By $f(x, y)$ we mean an expression in the variables x, y.

144. Surfaces of revolution. Consider the locus of the equation

(1) $$z = a(x^2 + y^2)$$

where a is positive. The equation in z, x coördinates of the section of the surface by the zx-plane, is obtained by substituting the equation of that plane, $y = 0$, in (1); the resulting equation is that of the parabola $z = ax^2$. The section by a plane $z = z_1$ parallel to the xy-plane, where z_1 is positive, has an equation obtained by substituting $z = z_1$ in (1); it is a circle

$$z = z_1, \quad x^2 + y^2 = \frac{z_1}{a}.$$

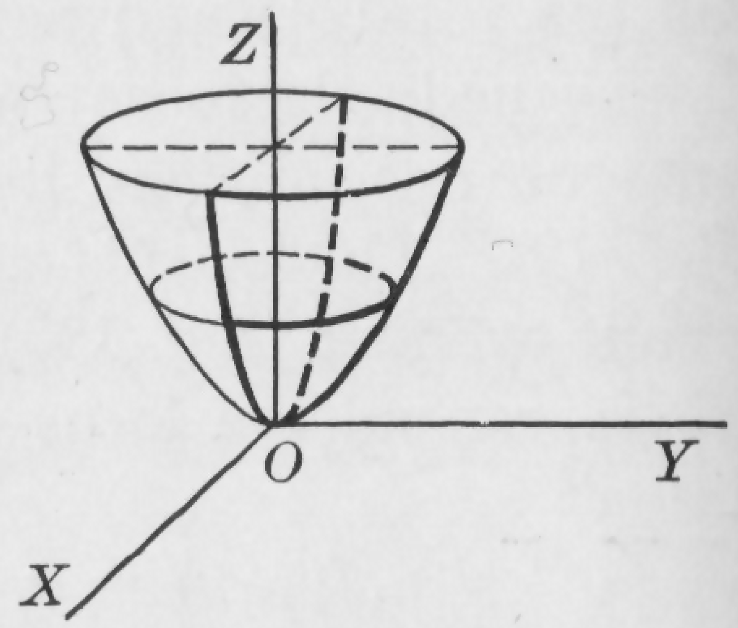


Fig. 137

The surface could be generated by rotating the parabola $z = ax^2$ about the z-axis. It is called a **paraboloid of revolution** (Fig. 137).

Consider now the locus of an equation of the general form

(2) $$f(z, \sqrt{x^2 + y^2}) = 0.$$

The section of this locus by the zx-plane, $y = 0$, is the plane curve

(3) $$f(z, x) = 0, \quad y = 0.$$

The section by a plane $z = z_1$ parallel to the xy-plane, is the plane locus

$$z = z_1, \quad f(z_1, \sqrt{x^2 + y^2}) = 0.$$

If this last equation is solved for $\sqrt{x^2 + y^2}$, we get an equation, or a set of equations, of the form $\sqrt{x^2 + y^2} =$ constant, hence its locus is a circle with its center on the z-axis, or a set of circles with centers on the z-axis. The locus of (1) is therefore the surface generated by rotating the plane curve (3) about the z-axis.

By interchanging x, y, and z in (1) we obtain the equations of surfaces of revolution about the other axes.

EXERCISES

Describe the locus in space of each of the following equations.

1. $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$
2. $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1.$
3. $y^2 - z^2 = 0.$
4. $xz = a^2.$
5. $z = \sin y.$
6. $z = \log x.$
7. $\frac{x^2 + z^2}{a^2} + \frac{y^2}{b^2} = 1.$
8. $\frac{x^2}{a^2} - \frac{y^2 + z^2}{b^2} = 1.$
9. $x^2 + y^2 = 4az.$
10. $\frac{x^2 + z^2}{a^2} - \frac{y^2}{b^2} = 1.$
11. $x^2 + y^2 = \sin^2 z.$
12. $y^2 + z^2 = x^2.$

Find an equation of the surface of revolution obtained by rotating each of the following plane curves as indicated.

13. $y^2 = 2px$ about the x-axis.
14. $\frac{x^2}{a^2} - \frac{z^2}{b^2} = 1$ about the x-axis.
15. $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ about the y-axis.
16. $\frac{x^2}{a^2} + \frac{z^2}{b^2} = 1$ about the z-axis.
17. $y = mx$ about the y-axis.
18. $y = \sin x$ about the x-axis.

145. Surfaces. If an equation

(1) $$z = f(x, y)$$

is such that a real value of z corresponds to each point (x, y) of a region R of the xy-plane, possibly including the whole plane, then the locus is clearly a surface. Likewise, as noted on page 315, an implicit equation

(2) $$f(x, y, z) = 0$$

will have a surface for its locus in the general case in which the equation determines one or more real values of z for each point (x, y) in a region R of the xy-plane. In the latter state-

ment the letter z may be interchanged with x or y. We shall consider a number of such equations in the following articles.

The appearance of the surface can be ascertained by finding its *traces* on the coördinate planes, that is, the curves of intersection with those planes, and by finding the intersections with planes parallel to the coördinate planes. It is sometimes useful to test for symmetry of the surface with respect to the coördinate planes and lines, and with respect to the origin (see Exercises 9 and 10, page 285). In some cases it is desirable to find the curves of intersection with planes other than those just mentioned. These procedures are illustrated in the next few articles, where we discuss the loci of certain second degree equations, the so-called *quadric surfaces.*

146. The ellipsoid. The locus of the equation

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$$

is called an **ellipsoid.*** Its trace on the xy-plane, where $z = 0$, is the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, \qquad z = 0.$$

Likewise the traces on the other coördinate planes are ellipses.

The intersection with a plane $z = z_1$, parallel to the xy-plane, is, if $z_1^2 < c^2$, the plane curve

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 - \frac{z_1^2}{c^2}, \qquad z = z_1.$$

The axes of this ellipse are of lengths $2a\sqrt{c^2 - z_1^2}/c$ and $2b\sqrt{c^2 - z_1^2}/c$, which decrease as z_1^2 increases; there is no real intersection when $z_1^2 > c^2$.

Intersections with planes parallel to the other coördinate planes are also ellipses.

* We assume that a, b, c are positive numbers.

The surface is symmetrical to the coördinate planes, since the equation is not changed by changing the sign of any one of the coördinates. The surface is shown in Figure 138.

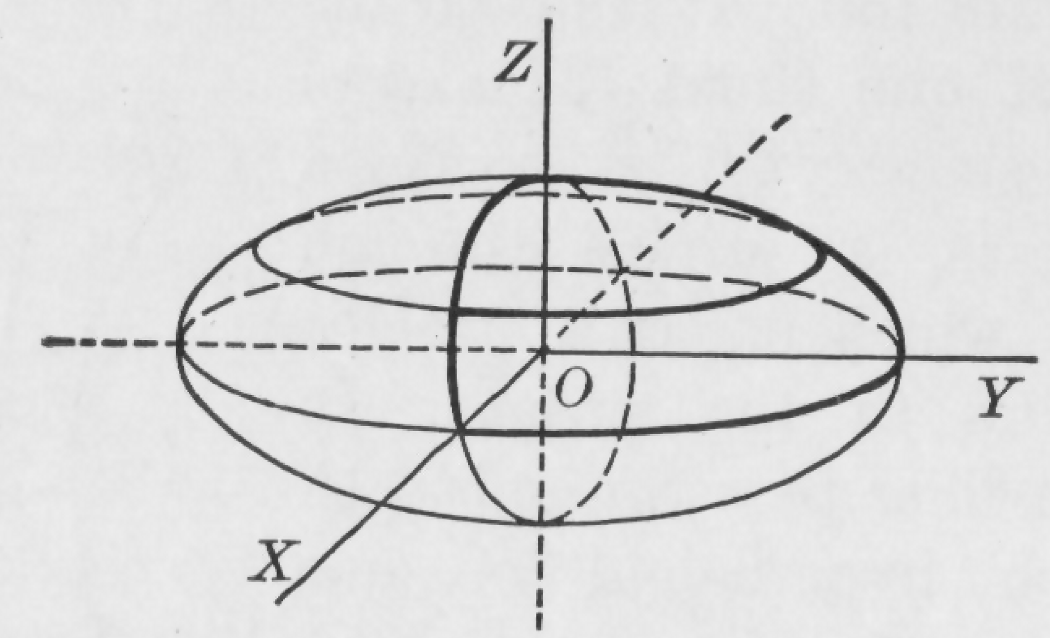


FIG. 138

If $a = b > c$, the ellipsoid can be generated by rotating the ellipse of the zx-plane

$$\frac{x^2}{a^2} + \frac{z^2}{c^2} = 1$$

about its minor axis. The surface is an **oblate spheroid.**

If $a > b = c$ the ellipsoid can be generated by rotating the ellipse about its major axis; it is a **prolate spheroid.**

147. The hyperboloids. The trace on the xy-plane of the surface

$$(1) \qquad \frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$$

is an ellipse, on the zx-plane a hyperbola, and on the yz-plane a hyperbola. The plane $z = z_1$ cuts the surface in the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 + \frac{z_1^2}{c^2}, \qquad z = z_1.$$

The axes of this ellipse are of lengths $2a\sqrt{c^2 + z_1^2}/c$ and $2b\sqrt{c^2 + z_1^2}/c$, which increase as z_1 increases. The plane $y = y_1$ cuts the surface in the hyperbola

$$\frac{x^2}{a^2} - \frac{z^2}{c^2} = 1 - \frac{y_1^2}{b^2}, \qquad y = y_1,$$

for which the transverse axis is on the x-axis if $y_1^2 < b^2$, and on the z-axis if $y_1^2 > b^2$.

The surface is symmetrical to the coördinate planes; it is shown in Figure 139. It is called the **hyperboloid of one sheet.** This surface can be shown to be a **ruled surface,** that is, a surface through each point of which passes a straight line which lies in the surface. In fact two such lines pass through each point of the hyperboloid of one sheet; for example, the plane $y = b$ intersects this surface in the two lines through $(0, b, 0)$,

$$\frac{x^2}{a^2} - \frac{z^2}{c^2} = 0, \qquad y = b.$$

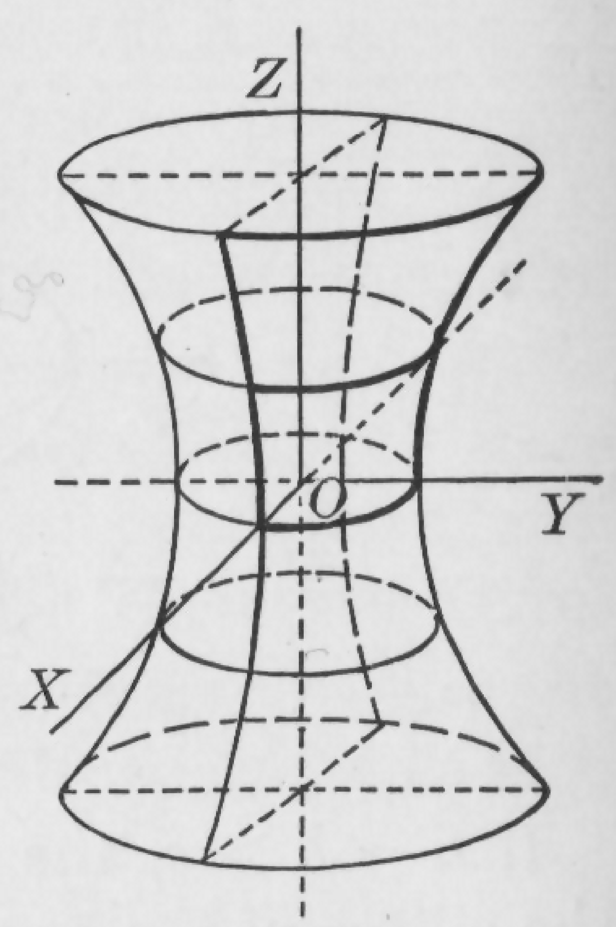


FIG. 139

The trace on the xz-plane of the surface

$$(2) \qquad \frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = -1$$

is the hyperbola

$$\frac{x^2}{a^2} - \frac{z^2}{c^2} = -1.$$

The trace on the yz-plane is the hyperbola,

$$\frac{y^2}{b^2} - \frac{z^2}{c^2} = -1,$$

but there is no trace on the xy-plane, since the locus of

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = -1$$

is imaginary.

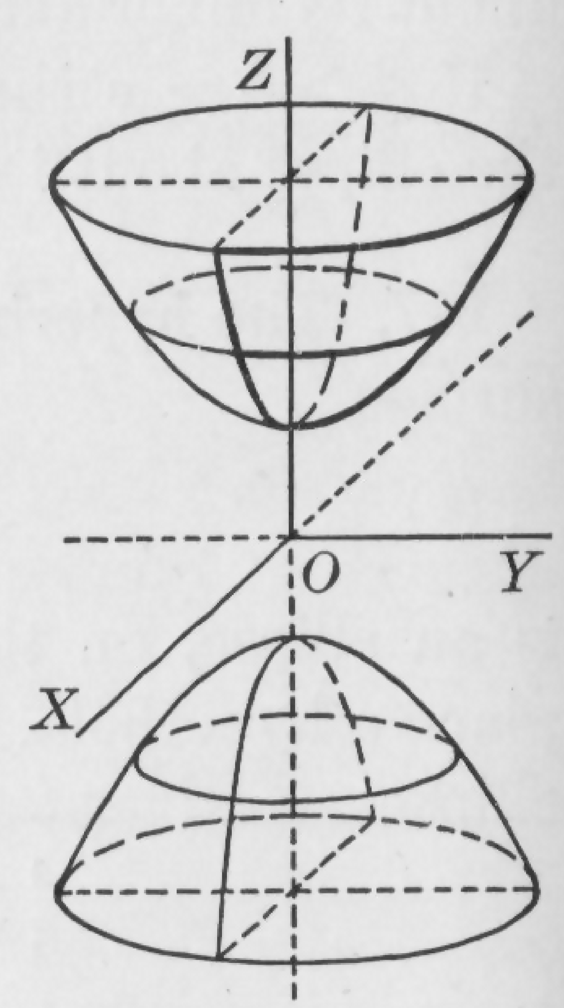


FIG. 140

A discussion similar to that for equation (1) indicates that the surface is as shown in Figure 140. It is known as the **hyperboloid of two sheets.**

148. The elliptic paraboloid. The traces of the **elliptic paraboloid**

$$(1) \qquad z = \frac{x^2}{a^2} + \frac{y^2}{b^2}$$

on the zx-plane and yz-plane are the parabolas

$$z = \frac{x^2}{a^2}, \quad y = 0; \qquad z = \frac{y^2}{b^2}, \quad x = 0.$$

The section of the surface by a plane $z = z_1$, where $z_1 > 0$, is the ellipse

$$\frac{x^2}{a^2z_1} + \frac{y^2}{b^2z_1} = 1, \qquad z = z_1.$$

The lengths of the axes of this ellipse increase indefinitely with z_1.

The surface is symmetrical with respect to the yz- and zx-planes, since the equation is unchanged by changing the signs of x and of y respectively, but it is not symmetrical with respect to the xy-plane. In fact there are no points on the surface below the xy-plane (Fig. 141).

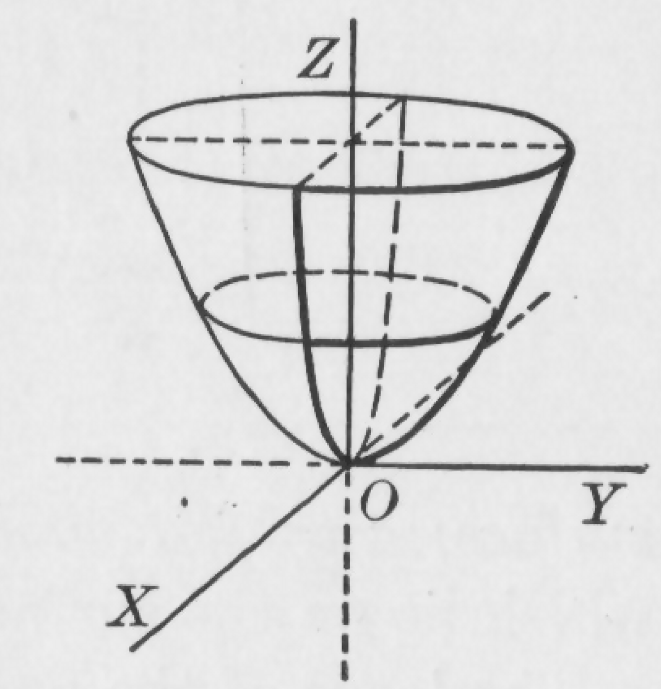


FIG. 141

149. The hyperbolic paraboloid. If we change a sign in equation (1) of the preceding section, we have the equation

$$(1) \qquad z = -\frac{x^2}{a^2} + \frac{y^2}{b^2}.$$

The locus is a surface which is cut by the zx-plane in a parabola, $z = -x^2/a^2$, $y = 0$, whose axis is the negative axis of z. It is cut by the yz-plane in a parabola $z = y^2/b^2$, $x = 0$, whose axis is the positive axis of z. The xy-plane cuts the surface in two straight lines whose equations are

$$-\frac{x^2}{a^2} + \frac{y^2}{b^2} = 0, \qquad z = 0;$$

these lines intersect at the origin. The plane $z = z_1$ cuts the surface in a hyperbola

$$-\frac{x^2}{a^2} + \frac{y^2}{b^2} = z_1, \qquad z = z_1.$$

If $z_1 > 0$ the transverse axis is parallel to the y-axis; if $z_1 < 0$ it is parallel to the x-axis. The locus is called a **hyperbolic paraboloid** (Fig. 142). It can be shown that this is a *ruled*

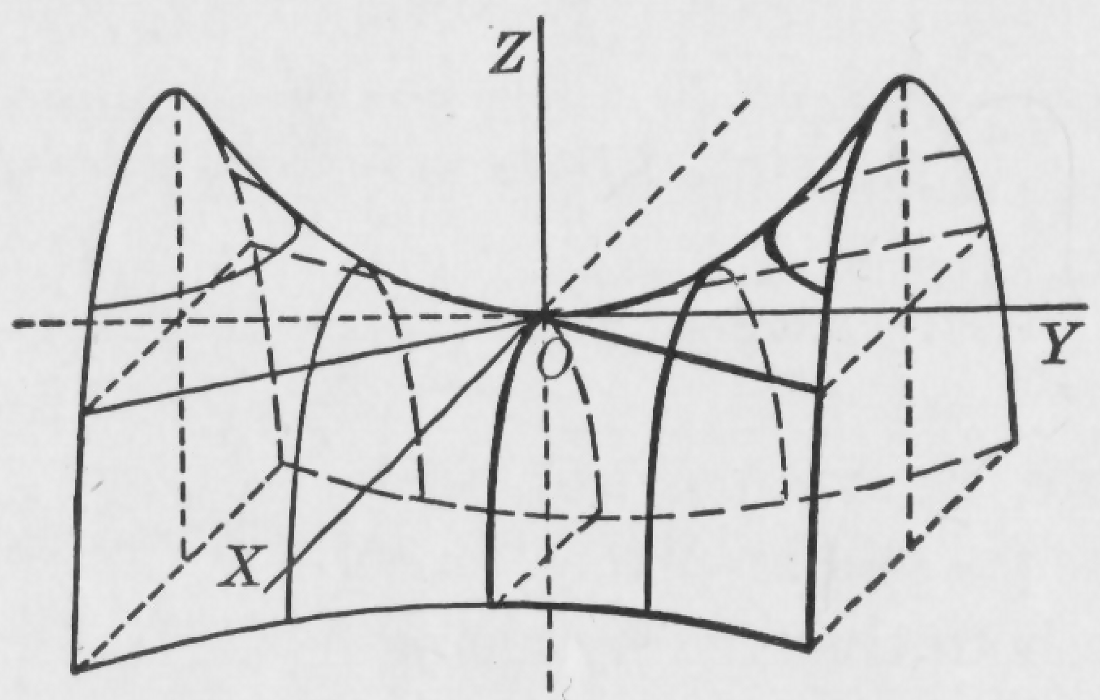


FIG. 142

surface; through every point of it pass two straight lines which lie on the surface.

The locus of the equation

(2) $$z = kxy$$

is also a hyperbolic paraboloid. To prove this rotate the x- and y-axes about the z-axis through an angle $-\pi/4$ into new x'- and y'-axes. The transformation is

$$x = \frac{x' + y'}{\sqrt{2}}, \qquad y = \frac{-x' + y'}{\sqrt{2}}.$$

Equation (2) becomes

$$z = -\frac{kx'^2}{2} + \frac{ky'^2}{2}$$

which is a special case of (1).

150. The cone. The locus of the equation

(1) $$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 0$$

is cut by the zx-plane in two straight lines which pass through the origin, O, their equations being

$$\frac{x^2}{a^2} - \frac{z^2}{c^2} = 0, \qquad y = 0.$$

We shall show that any plane whatever which contains the z-axis likewise cuts the locus in two straight lines which pass through O. To do this, we rotate the x- and y-axes about the z-axis through an angle θ. The new coördinates of a point being x', y', z, the relation between x, y and x', y' is given by the transformation (page 147)

$$x = x' \cos\theta - y' \sin\theta,$$
$$y = x' \sin\theta + y' \cos\theta.$$

Substituting in (1), and then setting $y' = 0$, we obtain the intersection of the zx'-plane with the locus; we have

$$x'^2\left(\frac{\cos^2\theta}{a^2} + \frac{\sin^2\theta}{b^2}\right) - \frac{z^2}{c^2} = 0,$$
$$y' = 0,$$

which are the equations of two straight lines through O. Since θ may be chosen so that the zx'-plane is any plane whatever containing the z-axis, we have established the proposition which we were to prove.

A plane perpendicular to the z-axis, $z = z_1$, cuts the locus in the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = \frac{z_1^2}{c^2}, \qquad z = z_1.$$

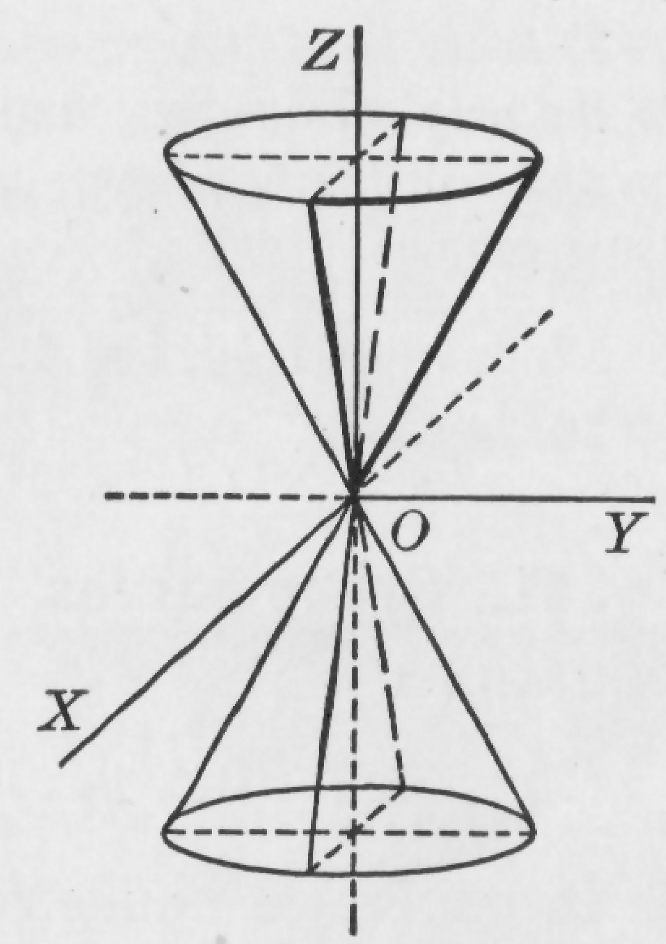


FIG. 143

The locus of (1) is an **elliptic cone** (Fig. 143) if $a \neq b$. If $a = b$ it is a **right circular cone.**

EXERCISES

Describe the locus of each of the following equations, and discuss its traces on the coördinate planes.

1. $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} + \dfrac{z^2}{c^2} = 1.$ **2.** $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} - \dfrac{z^2}{c^2} + 1 = 0.$

3. $x = \dfrac{y^2}{b^2} + \dfrac{z^2}{c^2}.$ **4.** $y = \dfrac{x^2}{a^2} - \dfrac{z^2}{c^2}.$

5. $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} - \dfrac{z^2}{c^2} = 0.$ **6.** $\dfrac{x^2 + y^2}{a^2} + \dfrac{z^2}{b^2} = 1.$

7. $\dfrac{x^2 + y^2}{a^2} - \dfrac{z^2}{b^2} = 1.$ **8.** $\dfrac{(x + y)^2}{a^2} - \dfrac{z^2}{b^2} = 0.$

9. $4x^2 + 4y^2 + 9z^2 = 36.$ **10.** $x^2 = 4(y^2 - z^2).$

11. $x^2 + 4y^2 = z^2 + 9.$ **12.** $x = y^2 - z^2.$

Find, by rotating axes in the xy*-plane, the type of curve in which each of the following surfaces is cut by every plane through the* z*-axis.*

13. $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} + \dfrac{z^2}{c^2} = 1.$ **14.** $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} - \dfrac{z^2}{c^2} = 1.$

15. $-\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} + \dfrac{z^2}{c^2} = 1.$ **16.** $z = ax^2 + by^2.$

Discuss the following surfaces, specifying their symmetry with respect to the origin, the axes, and the coördinate planes. Describe their traces on the coördinate planes, and their intersections with planes parallel to those planes.

17. $x^{\frac{2}{3}} + y^{\frac{2}{3}} + z^{\frac{2}{3}} = 1.$ **18.** $x^2 + y^2 + z^4 = 25.$

19. $xy = z^2.$ **20.** $xyz = 1.$

151. Space curves. The locus of a pair of simultaneous equations

$$(1) \qquad f(x, y, z) = 0, \quad \phi(x, y, z) = 0,$$

is in general a space curve, the curve of intersection of the two surfaces whose equations are (1). Thus the locus of the pair of equations

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1, \quad z = 0$$

is an ellipse in the xy-plane; the locus of the pair of equations

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1, \quad x^2 + y^2 + z^2 = R^2,$$

is the curve of intersection of an ellipsoid and a sphere, and is in general not a plane curve.

The curve (1) lies on every surface

$$k_1 f(x, y, z) + k_2 \phi(x, y, z) = 0,$$

where k_1 and k_2 are any constants not both zero.

A space curve may be described by a set of three parametric equations

$$(2) \qquad x = f_1(t), \quad y = f_2(t), \quad z = f_3(t),$$

with t as parameter. Corresponding to each value of t, perhaps limited to some range of values, there is a point (x, y, z), and as t varies the point describes a curve in space. The curve lies on every surface whose equation, $F(x, y, z) = 0$, is obtained by eliminating t from equations (2).

An illustration is furnished by the **helix,** the curve of motion of a point $P(x, y, z)$ which starts from the position $(a, 0, 0)$ and whose projection M on the xy-plane describes a circle with uniform angular velocity ω, the point P receding from the xy-plane at a uniform rate b, so that $z = bt$. At the end of time t the radius vector OM makes the angle ωt with the x-axis, and we have $x = a \cos \omega t$, $y = a \sin \omega t$. Thus the desired parametric equations of the helix are

$$(3) \qquad x = a \cos \omega t, \quad y = a \sin \omega t, \quad z = bt.$$

This curve lies on the cyclinder $x^2 + y^2 = a^2$.

If we modify the motion of P in the preceding illustration so that M starts at the origin and recedes from it at a uniform rate c while continuing to revolve about the origin as before, equations (3) are replaced by the following:

$$(4) \qquad x = ct \cos \omega t, \quad y = ct \sin \omega t, \quad z = bt.$$

From these equations we find that

$$\frac{x^2}{c^2} + \frac{y^2}{c^2} = \frac{z^2}{b^2}.$$

This last is the equation of a right circular cone on which the curve (4) lies. Hence this modified helix is called a **conical helix.**

152. Projection of a curve. Let the equations

(1) $$f(x, y, z) = 0, \quad \phi(x, y, z) = 0,$$

or

(1′) $$x = f_1(t), \quad y = f_2(t), \quad z = f_3(t),$$

define a curve C in space. If we eliminate z, or z and t, we obtain an equation

(2) $$F(x, y) = 0$$

which is satisfied by all points of C, and is an equation of a cylinder with elements parallel to the z-axis. The curve C lies on this cylinder.

To project a point of the curve C on the xy-plane, a line is drawn through the point parallel to the z-axis. This line therefore lies on the cylinder (2). The curve obtained by projecting all points of C on the xy-plane is therefore the curve (2) (or a part of it) considered as a plane curve in x, y coördinates.

Thus the projection of the helix

$$x = a \cos \omega t, \quad y = a \sin \omega t, \quad z = bt,$$

on the xy-plane is the circle

$$x^2 + y^2 = a^2;$$

on the zx-plane the curve

$$x = a \cos \frac{\omega z}{b};$$

and on the yz-plane the curve

$$y = a \sin \frac{\omega z}{b}.$$

The projection of the conical helix

$$x = ct \cos \omega t, \quad y = ct \sin \omega t, \quad z = bt,$$

on the xy-plane is found to be

$$x = \frac{cx}{\omega\sqrt{x^2 + y^2}} \tan^{-1} \frac{y}{x}.$$

EXERCISES

Describe each of the following space curves, giving its projections on each of the coördinate planes.

1. $x = t^2, \quad y = 2t, \quad z = t.$

2. $x = a \cos \theta, \quad y = b \sin \theta, \quad z = c\theta.$

3. $x^2 + y^2 + z^2 = 4a^2, \quad x^2 + y^2 - z^2 = a^2.$

4. $x^2 + y^2 = z^2, \quad x + y = z.$

5. $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} + \dfrac{z^2}{c^2} = 1, \quad x^2 + y^2 + z^2 = R^2.$

6. $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} - \dfrac{z^2}{c^2} = 1, \quad x = y.$

Prove that each of the following sets of simultaneous equations represents a plane curve.

7. $x = t - 2, \quad y = t + 2, \quad z = t^3.$

8. $x = \cos^2 \theta, \quad y = \sin \theta, \quad z = \sin^2 \theta.$

9. $x^2 + y^2 + z^2 = a^2, \quad x^2 + y^2 - z^2 = a^2.$

10. $x^2 - z^2 - y = 0, \quad x^2 - z^2 - y^2 = 0$ (two plane curves).

For proj – eliminate one variable

CHAPTER XVII

SYSTEMS OF COÖRDINATES

153. Translation of axes. If the coördinate axes are moved so that the new origin has coördinates (x_0, y_0, z_0) with respect to the first axes, the directions of the axes remaining unchanged, the old coördinates (x, y, z) of a point P are expressed in terms of the new coördinates (x', y', z') by the equations

$$\mathbf{x} = \mathbf{x}' + \mathbf{x}_0, \quad \mathbf{y} = \mathbf{y}' + \mathbf{y}_0,$$
$$\mathbf{z} = \mathbf{z}' + \mathbf{z}_0.$$

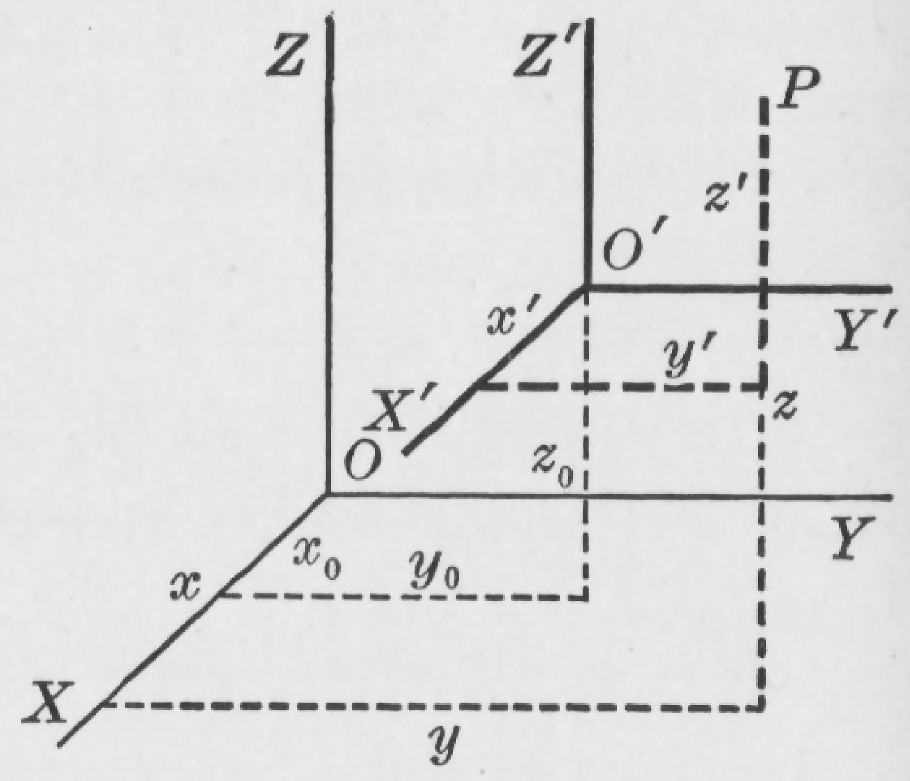


FIG. 144

By means of such a translation of axes we are sometimes able to simplify an equation and more readily to determine its locus; this is illustrated in the following example.

Example. — Determine the locus of the equation

$$x^2 - 4y^2 - 9z^2 + 8y - 36z = 56.$$

Solution. — We write the equation

$$x^2 - 4(y^2 - 2y + 1) - 9(z^2 + 4z + 4) = 16,$$

and translate the axes, taking the new origin at $(0, 1, -2)$. We have

$$x = x', \quad y = y' + 1, \quad z = z' - 2,$$

and the equation becomes

$$\frac{x'^2}{16} - \frac{y'^2}{4} - \frac{z'^2}{16/9} = 1.$$

The locus is therefore a hyperboloid of two sheets, with center at $(0, 1, -2)$ and semi-axes of lengths 4, 2, and 4/3 respectively. The x'-axis cuts the surface.

154. General rotation of axes. Let us consider two systems of rectangular coördinates with the same origin O. The coördinates of a point P with respect to the first system will be called x, y, z, and with respect to the second system x', y', z'.

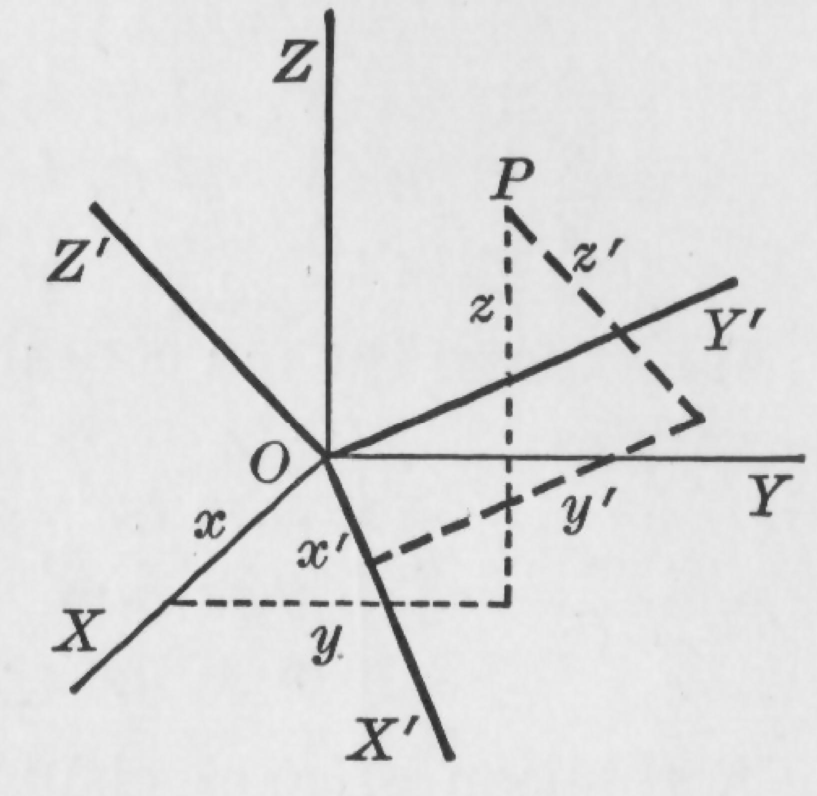


FIG. 145

Let the direction cosines of the x'-axis with respect to the x-, y-, and z-axes be l_1, m_1, n_1. The positive x'-axis is the normal to the $y'z'$-plane; hence the equation of this plane in the x, y, z system is (page 298)

$$l_1x + m_1y + n_1z = 0.$$

The coördinate x' is the distance of P from the $y'z'$-plane; hence, by the formula (page 298) for the distance from a plane to a point,

$$x' = l_1x + m_1y + n_1z.$$

We obtain similar equations for y' and z'. If l_2, m_2, n_2 and l_3, m_3, n_3 are the direction cosines of the y'-axis and z'-axis, we thus have

$$\text{(1)} \qquad \begin{aligned} \mathbf{x}' &= \mathbf{l}_1\mathbf{x} + \mathbf{m}_1\mathbf{y} + \mathbf{n}_1\mathbf{z}, \\ \mathbf{y}' &= \mathbf{l}_2\mathbf{x} + \mathbf{m}_2\mathbf{y} + \mathbf{n}_2\mathbf{z}, \\ \mathbf{z}' &= \mathbf{l}_3\mathbf{x} + \mathbf{m}_3\mathbf{y} + \mathbf{n}_3\mathbf{z}. \end{aligned}$$

These three equations are the equations of transformation for a **general rotation of axes.** The nine direction cosines satisfy several equations. First, we have

$$\begin{aligned} l_1^2 + m_1^2 + n_1^2 &= 1, \\ l_2^2 + m_2^2 + n_2^2 &= 1, \\ l_3^2 + m_3^2 + n_3^2 &= 1. \end{aligned}$$

Since the axes are mutually perpendicular, it follows that

$$\begin{aligned} l_1l_2 + m_1m_2 + n_1n_2 &= 0, \\ l_1l_3 + m_1m_3 + n_1n_3 &= 0, \\ l_2l_3 + m_2m_3 + n_2n_3 &= 0. \end{aligned}$$

Furthermore the direction cosines of the x-, y-, and z-axes with respect to the x'-, y'-, z'-axes are l_1, l_2, l_3, and m_1, m_2, m_3, and n_1, n_2, n_3 respectively. Hence we have

$$\begin{aligned} l_1^2 + l_2^2 + l_3^2 &= 1, & l_1m_1 + l_2m_2 + l_3m_3 &= 0,\\ m_1^2 + m_2^2 + m_3^2 &= 1, & l_1n_1 + l_2n_2 + l_3n_3 &= 0,\\ n_1^2 + n_2^2 + n_3^2 &= 1, & m_1n_1 + m_2n_2 + m_3n_3 &= 0. \end{aligned}$$

We express the old coördinates in terms of the new by the equations

$$\begin{aligned} x &= l_1x' + l_2y' + l_3z',\\ y &= m_1x' + m_2y' + m_3z',\\ z &= n_1x' + n_2y' + n_3z'. \end{aligned}$$

A rotation of axes changes an equation $f(x, y, z) = 0$ of degree n into an equation $F(x', y', z') = 0$ of the same degree. This property may be used to prove the theorem that *every plane section of a quadric surface is a conic section* in the general sense of Chapter XII, *or a straight line;* for we may take the cutting plane as the new $x'y'$-plane; the new equation of the quadric surface is still of the second degree, and its section by the plane $z' = 0$ has an equation of not more than the second degree in x', y'. In particular this proves the statement of page 110 that *every plane section of a right circular cone is a conic section.*

EXERCISES

By translation of axes determine the loci of each of the following equations 1–6.

1. $x^2 + 25y^2 + z^2 - 2x + 50y = 10.$
2. $4x^2 - y^2 - 9z^2 + 4y + 18z = 4.$
3. $x^2 + 4y^2 - 9z^2 + 8x + 36z = 21.$
4. $x^2 + 4z^2 - 2x - y = 0.$
5. $x^2 - 4y^2 + 8y + z = 0.$
6. $4x^2 + 4y^2 - z^2 - 2z = 1.$
7. Show by actual substitution of new coördinates in place of the old that the equation of a sphere whose center is at the origin has the same form after a general rotation of axes.

8. Find the equations of the general rotation of axes for which $l_1 = \frac{1}{3}$, $m_1 = \frac{2}{3}$, n_1 is negative, $l_2 = \frac{2}{3}$, $m_2 = \frac{1}{3}$.
9. Find the general rotation of axes such that the new positive x'-axis passes through the point whose old coördinates were $(1, 1, \sqrt{2})$, the y'-axis through $(1, 1, -\sqrt{2})$, and the z'-axis through $(1, -1, 0)$. Use this transformation to prove that the surface $x^2 + y^2 + 2z^2 + 2xy = 8$ is a cylinder.
10. Find the equations of the general rotation of axes for which the $x'y'$-, $y'z'$-, $z'x'$-planes are respectively

$$x + 2y + 2z = 0, \quad 2x + y - 2z = 0, \quad 2x - 2y + z = 0.$$

11. Use the rotation of axes described in Exercise 9 to discuss the section of the cone $x^2 + y^2 = \left(z - \frac{1}{\sqrt{2}}\right)^2$ by the plane $y = x$.

155. Cylindrical coördinates. Thus far in solid analytic geometry we have employed only rectangular coördinates. Another useful system is that of **cylindrical coördinates,** in which the z-coördinate is defined as in the rectangular system, but x, y are replaced by polar coördinates r, θ in the xy-plane. Thus in Figure 146, where M is the foot of the perpendicular from the point $P(x, y, z)$ to the xy-plane, cylindrical coördinates of P are (r, θ, z) as shown. We have

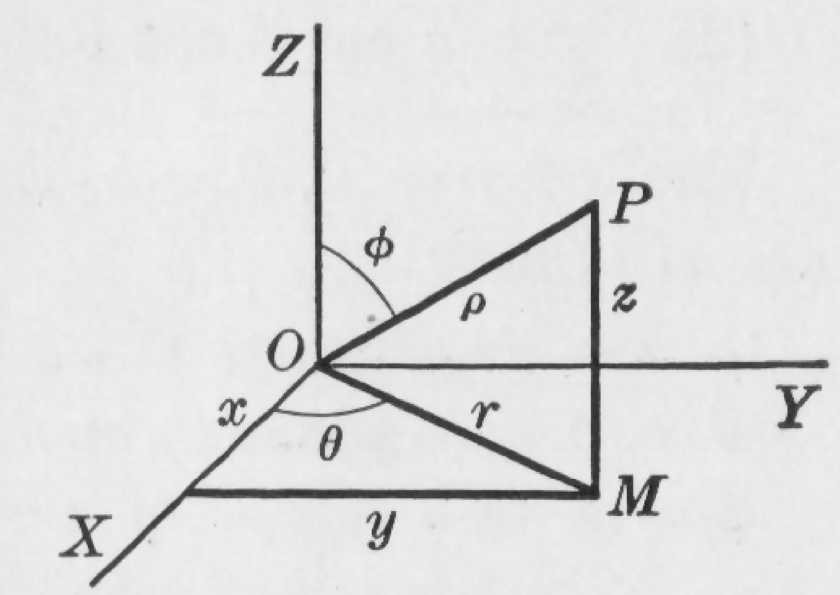


FIG. 146

$$r = \overline{OM}, \qquad \theta = \angle XOM, \qquad z = MP.$$

The equations which express the rectangular in terms of the cylindrical coördinates for Figure 146 are

$$(1) \qquad x = r\cos\theta, \quad y = r\sin\theta, \quad z = z.$$

The loci $r = a$, $\theta = b$, $z = c$, where a, b, c are constants, are as follows:

r = a *is a right circular cylinder about the* z*-axis.*
θ = b *is a plane through the* z*-axis,*
z = c *is a plane perpendicular to the* z*-axis.*

156. Spherical coördinates. Another system, that of **spherical** (or **polar** or **geographical**) **coördinates,** is also illustrated in Figure 146. Here the spherical coördinates of $P(x, y, z)$ are * (ρ, θ, ϕ) where

$$\rho = \overline{OP}, \qquad \angle\, XOM = \theta, \qquad \angle\, ZOP = \phi.$$

For a point P on the surface of a sphere $\rho =$ constant, we call θ the **longitude** and ϕ the **co-latitude** of P.

The spherical and cylindrical coördinates of P are related as follows:

$$(1) \quad \rho = \sqrt{r^2 + z^2}, \qquad \theta = \theta, \qquad \cos\phi = \frac{z}{\sqrt{r^2 + z^2}},$$

$$r = \rho \sin\phi, \qquad z = \rho\cos\phi.$$

The following relations hold between rectangular and spherical coördinates:

$$(2) \quad x = \rho\sin\phi\cos\theta, \quad y = \rho\sin\phi\sin\theta, \quad z = \rho\cos\phi.$$

The loci $\rho = a$, $\theta = b$, $\phi = c$, where a, b, c are constants, are as follows:

$\rho =$ a *is a sphere whose center is at* O, if a is positive.
$\theta =$ b *is a plane through the* z-*axis.*
$\phi =$ c *is a nappe of a right circular cone whose axis is the* z-*axis and whose vertex is at* O.

EXERCISES

Describe the locus of each of the following equations, or sets of equations, in cylindrical coördinates; transform to rectangular coördinates.

1. $r = a\cos\theta$.
2. $r = z$.
3. $r^2 + z^2 = a^2$.
4. $r^2 = a^2\cos 2\theta$.
5. $r = a, \quad \theta = b$.
6. $r = a, \ z = b$.
7. $r = a\sin\theta, \ z = b$.
8. $r = a, \ \theta = bt, \ z = ct$.

* ρ is the Greek letter "rho."

Describe the locus of each of the following equations, or sets of equations, in spherical coördinates; transform to rectangular coördinates.

9. $\rho = a\cos\phi$.
10. $\rho\sin\phi = a$.
11. $\rho^2\cos 2\phi = a^2$.
12. $\rho = a\sin\theta\cos\phi$.
13. $\rho = a, \ \theta = b$.
14. $\rho = a, \ \phi = c$.
15. $\theta = b, \ \phi = c$.
16. $\rho = a, \ \rho^2\cos 2\phi = a^2$.
17. $\rho = at, \ \theta = bt, \ \phi = c$.
18. $\rho\sin\theta = a, \ \rho\cos\theta = bt, \ \theta = ct$.

Express each of the following equations in cylindrical and in spherical coördinates.

19. $x^2 + y^2 + z^2 = a^2$.
20. $x^2 + y^2 = a^2$.
21. $z = x^2 + y^2$.
22. $z = x$.
23. $\dfrac{x^2 + y^2}{a^2} + \dfrac{z^2}{c^2} = 1$.
24. $\dfrac{x^2 - y^2}{a^2} - \dfrac{z^2}{c^2} = 1$.

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NUMBERS REFER TO PAGES

ANSWERS

Answers to even-numbered problems are not given here. Answers to odd-numbered problems are omitted in a few cases where nothing would be left for the student to do if the answer were given. Where approximate results are required, numbers are given to one place after the decimal point except in Chapter XIII, and angles are given to the nearest number of degrees.

Page 15

5. (1, 2), (1, $-$ 2), ($-$ 1, $-$ 2). **7.** (*a*) First. (*b*) Third.

9. (*a*) A straight line parallel to the y-axis and five units to the left of that axis.

11. $y = -x$. **13.** (a, b), $(-a, b)$, or $(a, -b)$.

Page 20

1, 3, 5, 7, 9. Straight lines. **11, 13.** Parabolas.

15. Circle. **17.** Ellipse. **19, 21.** Hyperbolas.

23. Circle. **25.** Ellipse.

Page 22

1. $x = 4, y = 1$. **3.** $x = .5, y = -1.5$.

5. $x = 2, y = 1$; and $x = .5, y = -.5$.

7. $x = 3, y = -4$; and $x = -1.4, y = 4.8$.

9. $x = 2, y = 0$; and $x = 2.5, y = 1.5$.

11. $x = 3.2, y = 1.2$; $x = 3.2, y = -1.2$; $x = -3.2, y = 1.2$; and $x = -3.2, y = -1.2$.

13. $x = -3, y = 0$; $x = -1, y = 5.7$; and $x = -1, y = -5.7$.

Pages 29, 30

7. $A : (4, 405°), (-4, 225°), (-4, -135°)$, etc.
$B : (1, 320°), (-1, 140°), (-1, -220°)$, etc.
$C : (2, -90°), (-2, 90°), (-2, -270°)$, etc.

9. $A(4.2, 4.2)$, $B(4.2, -4.2)$, $C(-4.2, -4.2)$, $D(4.2, 4.2)$.

11. $A(1.7, -4.7)$, $B(-3.8, 1.4)$, $C(0, -2)$, $D(0, -3)$.

13. $A(-3.5, 2)$, $B(4, 0)$, $C(4.2, 4.3)$, $D(2.5, -5.5)$.

15. $A : (2.8, 45°), (-2.8, 225°)$. $B : (1.4, 315°), (-1.4, 135°)$.
$C : (5, 127°), (-5, 307°)$. $D : (5, 233°), (-5, 53°)$.

17. $A : (3, 90°), (-3, 270°)$. $B : (3, 180°), (-3, 0°)$.
$C : (2.7, 292°), (-2.7, 112°)$. $D : (5.5, 235°), (-5.5, 55°)$.

19. $\tan\theta = 1$, or $\theta = 45°$.
21. $r\cos\theta = a$.
23. $r(a\cos\theta + b\sin\theta) = c$.
25. $r = \cos\theta$.
27. $r\sin^2\theta - 4\cos\theta = 0$.
29. $r^2\cos 2\theta = 16$.
31. $r^2\sin 2\theta = 5$.
33. $y = (\tan 1)x$, straight line.
35. $x^2 + y^2 = 100$, circle.
37. $x = 4$, straight line.
39. $x + 2y = 4$, straight line.
41. $x^2 + y^2 = 10y$, circle.
43. $2xy = 25$, hyperbola.
45. $x^2 + y^2 = 10(y - x)$, circle.

Pages 34, 35

1. $-1, 4$; $-8, -8$; $9, 4$.
3. $-1, -4$; $4, 4$; $-3, 0$.
5. $\sqrt{17}$; $8\sqrt{2}$; $\sqrt{97}$.
7. $\sqrt{17}$; $4\sqrt{2}$; 3.
17. Not a rectangle.
19. A rectangle (square).

Pages 38, 39

5. (*a*) 1, 45°; (*b*) -1, 135°; (*c*) 3, 71°; (*d*) $\frac{37}{23}$, 58°.

Pages 42, 43

1. 99°.
3. 100°.
5. $A = 45°, B = 90°, C = 45°$.
7. $A = 57°, B = 75°, C = 48°$.
9. $A = 99°, B = 49°, C = 32°$.
11. $m = -3$.
13. $m = -1$.
15. Not a rectangle.
17. A rectangle (square).
19. A rectangle (square).

Pages 46, 47

1. (*a*) $(-\frac{2}{3}, \frac{1}{3})$, (*b*) $(-\frac{4}{3}, \frac{5}{3})$, (*c*) $(2, -5)$, (*d*) $(2, -5)$.
5. $(-\frac{3}{2}, \frac{1}{2}), (-6, 5)$.
7. $(0, 2), (-2, 5)$.
9. $(\frac{5}{3}, -\frac{4}{3})$.
11. $\left(\dfrac{x_1 + x_2 + x_3}{3}, \dfrac{y_1 + y_2 + y_3}{3}\right)$.

Pages 50, 51

5. $-\frac{3}{2}, -\frac{3}{1}$.
9. 78°.
11. $(1, 4), (3, 8), (-1, -4)$.
13. $a = \frac{1}{5}, b = \frac{12}{5}$.
15. $2, -\frac{1}{2}$.
17. $(\frac{35}{17}, \frac{47}{17})$.

Pages 55, 56

1. (*a*) $x = -3$.
3. (*a*) $x + y = 0$; (*b*) $2x - 3y - 2 = 0$;
(*c*) $5x + 4y + 8 = 0$; (*d*) $x - 2y - 5 = 0$.
5. (*a*) $x - y = 0$; (*b*) $\sqrt{3}x - y + 2 = 0$;
(*c*) $x + y + 2 = 0$; (*d*) $\sqrt{3}x + y - 2 = 0$.
7. (*a*) $y = 0$; (*b*) $7x - 2y - 1 = 0$;
(*c*) $3x + 7y + 15 = 0$; (*d*) $x = 0$.
9. (*a*) $\frac{1}{2}$; (*b*) $-\frac{3}{2}$; (*c*) 1; (*d*) 0.
11. (*a*) $x - y + 2 = 0$; (*b*) $x + y + 2 = 0$.
17. (*a*) $x + y - 3 = 0$; (*b*) $x - y - 3 = 0$.

Pages 59, 60

1. (*a*) $y = -2$; (*b*) $y = -x$.
3. (Simplest forms) (*a*) $2x - y + 2 = 0$; (*b*) $3x + 2y + 6 = 0$;
(*c*) $x - 5y - 4 = 0$.
5. (*a*) $1, -1$; (*b*) $-1, -1$; (*c*) $1, -\frac{3}{2}$; (*d*) $-\frac{3}{2}, -\frac{5}{2}$.
7. (*a*) $1, -1$; (*b*) $-1, -1$; (*c*) $\frac{3}{2}, -\frac{3}{2}$; (*d*) $-\frac{5}{3}, -\frac{5}{2}$.
9. $y = m(x - a)$.
11. $x + y = 5$.
13. $\tan\theta = 7$.
15. $-\dfrac{b}{a}$.

Pages 67, 68

1. (*a*) $x = 0$; (*d*) $x + y + 2\sqrt{2} = 0$.
3. (*a*) 0°, 0; (*d*) 215°, 1.
5. (*a*) 0; (*b*) 1; (*c*) 1; (*d*) $\dfrac{4}{\sqrt{5}}$.
7. $5x + 12y - 169 = 0$.
9. Distance = 1.
11. (*a*) $\frac{12}{5}$; (*b*) $\dfrac{1}{\sqrt{2}}$.
13. (*a*) $\sqrt{5}$; $\frac{8}{13}\sqrt{13}$.

Pages 69, 70

1. (*a*) $r\sin\theta = -1$; (*b*) $\theta = \tan^{-1}\frac{1}{2}$; (*c*) $r\cos(\theta - 45°) = \sqrt{2}$.
3. (*a*) $r\cos\theta = 1$; (*b*) $\theta = \dfrac{3\pi}{4}$; (*c*) $r(3\cos\theta + 4\sin\theta) = 5$.
5. (*a*) $y = 2$; (*b*) $x - y = 0$; (*c*) $x + y - 5\sqrt{2} = 0$.
7. 45°, 1.
9. $r(\cos\theta - 3\sin\theta) = 1$.
11. -1.

Pages 70–72 (Miscellaneous Exercises)

1. $x + 4y - 10 = 0.$ **3.** $\frac{1}{1}.$

5. $(4, 3).$ **7.** $3x - 2y = 30.$

9. $0, -\frac{15}{8}.$ **11.** $(-\frac{1}{3}, 0).$

13. $x + 5y + 9 = 0,\quad 5x - y - 7 = 0.$

15. $x + y - 4\sqrt{2} = 0,\quad x + y + 4\sqrt{2} = 0.$

17. (*a*) $3x - 5y + 6 = 0,\quad x + 3y + 2 = 0,\quad 5x + y - 18 = 0;$
(*b*) $3x - 5y - 22 = 0,\quad x + 3y - 12 = 0,\quad 5x + y + 10 = 0;$
(*c*) $5x + 3y - 7 = 0,\quad 3x - y - 4 = 0,\quad 2x - 10y - 2 = 0.$

19. (*a*) $x - y - 1 = 0,\quad 4x - y - 3 = 0,\quad x - 4y - 2 = 0;$
(*b*) $x - y - 1 = 0,\quad 2x + 3y - 4 = 0,\quad 3x + 2y - 5 = 0.$

25. Center, $(1, 1)$; radius, 5. **27.** $90°, 45°, 45°.$

29. $3x - 5y - 21 = 0,\quad 5x + 3y - 1 = 0.$

Pages 79, 80

1. (*a*) $\frac{11}{5}$, opposite sides; (*b*) $\frac{6}{13}\sqrt{13}$, same side; (*c*) 1, same side.

3. $6, \frac{36}{5}, \frac{36}{5}.$

5. $14x - 112y + 365 = 0,\quad 64x + 8y - 235 = 0.$

7. $(2\sqrt{2} - \sqrt{5})x - (\sqrt{5} - \sqrt{2})y + 4\sqrt{2} + 5\sqrt{5} = 0.$

9. $2y + 5 = 0,\quad 2x + y = 0,\quad 2x - y - 5 = 0.$

11. $\frac{5}{2}.$ **13.** 16. **15.** 15. **17.** 2.

Pages 82, 83

3. $Ax + By = 0.$ **5.** $x + y + k = 0.$

7. $2x + y + k = 0.$ **11.** y-intercept $= 2.$

13. Slope $= 1.$ **15.** $x - y + 4 = 0.$

17. $x + y - 10 = 0, x - 4y = 0.$

19. $x + 2y + 10\sqrt{5} = 0,\quad x + 2y - 10\sqrt{5} = 0.$

Pages 85, 86

1. (*a*) $y + 2x - 3 = 0$; (*b*) $5x + 5y - 12 = 0.$

3. $5y - 9 = 0;\quad 5x - 3 = 0.$

5. $3x - 3y - 5 = 0,\quad x + y - 3 = 0.$

7. A parallel line whose distances from the given lines are proportional to the numerical values of k_2 and k_1, provided $A_1 = A_2$.

Page 92

1. (*c*) and (*e*) are parallel; (*a*) and (*d*) are perpendicular; (*b*) is perpendicular to (*c*) and (*e*).

Pages 92–94 (Miscellaneous Exercises)

1. (*a*) $x - 2y = 0$; (*b*) $2x + y = 0$; (*c*) $\frac{6}{5}\sqrt{5}.$

3. (*a*) $9x - y + 29 = 0$; (*b*) $x + 9y - 15 = 0$; (*c*) $\frac{33}{82}\sqrt{82}.$

5. (*a*) $\tan A = 3$; (*b*) $(2\sqrt{2} + \sqrt{5})x + (\sqrt{5} - \sqrt{2})y = 0;$
(*c*) area $= 6.$

7. (*a*) $\tan A = -\frac{11}{2};$
(*b*) $(2\sqrt{5} - 3)x + (4 + \sqrt{5})y - (17 - 4\sqrt{5}) = 0;$
(*c*) area $= \frac{33}{2}.$

9. (*a*) $(1 + 2\sqrt{2x}) + (3 + \sqrt{2})y - (18\sqrt{2} + 14) = 0;$
(*b*) $7x + y - 58 = 0$; (*c*) area $= 15.$

11. (*a*) $(3 + \sqrt{2})x + (2\sqrt{2} - 1)y + (9\sqrt{2} + 2) = 0;$
(*b*) $7x - 7y - 12 = 0$; (*c*) area $= \frac{72}{7}.$

13. $x - y = 0,\quad (\sqrt{2} + 1)x + y = \sqrt{2},\quad x + (\sqrt{2} + 1)y = \sqrt{2}.$

15. $x + 5y + 3 = 0,\quad 7x + 3y - 13 = 0,\quad 3x - y - 8 = 0.$

17. $(-3, 9),\quad (1, 1),\quad (15, 3);\quad$ area $= 60.$

19. Area $= \frac{371}{12}.$ **21.** On the same side.

23. $y + 2 = 0,\quad 7x - 24y + 50 = 0,\quad 4x - 3y - 10 = 0,$
$3x + 4y - 10 = 0.$

Pages 97, 98

1. (*a*) $x^2 + y^2 - 8y - 20 = 0$; (*b*) $x^2 + y^2 - 12x + 4y + 15 = 0;$
(*c*) $x^2 + y^2 + 6x - 8y - 1 = 0.$

3. (*a*) $(4, 0), 4$; (*b*) $(-2, 4), 5$; (*c*) $(3, -4)$, point circle;
(*d*) imaginary; (*e*) $(-\frac{5}{6}, -2), \frac{13}{6}.$

5. $x^2 + y^2 - 2x - 4y - 20 = 0.$

7. $x - 4y - 19 = 0.$ **9.** $(\frac{16}{13}, -\frac{28}{13}).$

Pages 100, 101

1. $x^2 + y^2 - 6x + 10y = 0.$

3. $4x^2 + 4y^2 - 79x - 32y + 190 = 0.$

5. $x^2 + y^2 + 8x - 6y = 0.$

7. $8x^2 + 8y^2 + 79x + 32y + 95 = 0.$

9. $x^2 + y^2 + 6x + 8y = 0.$

11. $360x^2 + 360y^2 - 120x - 1680y + 1441 = 0.$

13. $x^2 + y^2 - 4x - 4y + 4 = 0.$

15. $x^2 + y^2 - 2(7 - \sqrt{17})(x + y) + 18(5 - \sqrt{17}) = 0.$

17. $x^2 + y^2 + 6x + 8y - 56 = 0.$

19. $x^2 + y^2 + 6x - 16y + 48 = 0;$
$x^2 + y^2 - \frac{74}{121}x - \frac{336}{121}y - \frac{1392}{121} = 0.$

21. $x^2 + y^2 - 7x + 5y - 14 = 0.$

23. Center $(0, 0)$, radius $= 4$. 25. Center $(\frac{11}{5}, -\frac{2}{5})$, radius $= 5$.

Pages 104, 105

3. $4x^2 + 4y^2 - 9x - 25 = 0.$

5. (a) $4x + 1 = 0$; (b) $4x - 6y + 3 = 0$; (c) $8x - 15y = 4$.

9. $(\frac{8}{5}, \frac{29}{5})$.

Pages 105, 106

1. (a) $r^2 - 4r(\cos\theta + \sin\theta) + 4 = 0$;
(b) $r = 2\cos\theta + 2\sqrt{3}\sin\theta$; (c) $r = 4\cos\theta$.

7. $\left(4, \frac{\pi}{2}\right)$, 4. 9. $\left(5, \frac{3\pi}{2}\right)$, 5.

11. $\left(4, \frac{\pi}{4}\right)$, 4. 13. $(3, 0)$, 5.

15. $\left(10, -\frac{\pi}{3}\right)$, 10. 21. $\sqrt{25 - 12\sqrt{2}}$.

Pages 107–109

15. $4x^2 + 4y^2 - 42x + 9y = 0$. No.

19. $x^2 + y^2 + 3x - 9y = 0.$

Pages 114, 115

3. (a) $V(0, 0)$, $F(2, 0)$; (b) $V(0, 0)$, $F(-2, 0)$;
(c) $V(0, 0)$, $F(0, 1)$.

5. (a) $V(0, 0)$, $F(\frac{5}{2}, 0)$; (b) $V(0, 0)$, $F(0, -1)$;
(c) $V(0, 0)$, $F(-\frac{5}{2}, 0)$.

7. (a) $y^2 = 20x$; (b) $y^2 = 16x$;
(c) $x^2 = -16y$; (d) $x^2 = 40y$.

9. (a) $(y - 4)^2 = 8(x - 2)$; (b) $(y - 4)^2 = 16(x - 2)$.

11. $16x^2 - 24xy + 9y^2 + 72x + 96y - 144 = 0.$

13. $y^2 = -12x.$

Pages 121, 122

3. $\frac{x^2}{169} + \frac{y^2}{144} = 1.$ 5. $\frac{x^2}{400} + \frac{y^2}{256} = 1.$

7. $\frac{x^2}{16} + \frac{y^2}{25} = 1.$ 9. $\frac{x^2}{25} + \frac{y^2}{289} = 1.$

31. The ellipse which has the equation $\frac{x^2}{36} + \frac{y^2}{27} = 1.$

Pages 124, 125

1. $\frac{x^2}{a^2} + \frac{y^2}{a^2(1 - e^2)} = 1.$ 3. $\frac{4x^2}{225} + \frac{4y^2}{81} = 1.$

5. $\frac{x^2}{576} + \frac{169y^2}{14{,}400} = 1.$

7. $\frac{x^2}{64} + \frac{y^2}{49} = 1$, and $\frac{15x^2}{3136} + \frac{y^2}{49} = 1.$

9. $(\pm 4, 0)$, $\frac{4}{5}$, $x = \pm \frac{25}{4}$.

11. $(\pm \frac{25}{26}, 0)$, $\frac{5}{13}$, $x = \pm \frac{13}{2}$. 13. $(0, \pm 4)$, $\frac{4}{5}$, $y = \pm \frac{25}{4}$.

Pages 129, 130

5. $\frac{x^2}{3^2} - \frac{y^2}{2^2} = 1$; $F(\pm \sqrt{13}, 0)$; $V(\pm 3, 0)$; latus rectum, $\frac{8}{3}$; asymptotes, $y = \pm \frac{2}{3}x$.

7. $\frac{x^2}{6^2} - \frac{y^2}{6^2} = 1$; $F(\pm 6\sqrt{2}, 0)$; $V(\pm 6, 0)$; latus rectum, 12; asymptotes, $y = \pm x$.

9. $\frac{x^2}{10^2} - \frac{y^2}{2^2} = 1$; $F(\pm 2\sqrt{26}, 0)$; $V(\pm 10, 0)$; latus rectum, $\frac{4}{5}$; asymptotes, $y = \pm \frac{1}{5}x$.

11. $\frac{x^2}{(\frac{9}{8})^2} - \frac{y^2}{9^2} = 1$; $F(\pm \frac{9}{8}\sqrt{65}, 0)$; $V(\pm \frac{9}{8}, 0)$; latus rectum, 144; asymptotes, $y = \pm 8x$.

13. $\frac{y^2}{2^2} - \frac{x^2}{3^2} = 1$; $F(0, \pm \sqrt{13})$; $V(0, \pm 2)$; latus rectum, 9; asymptotes, $y = \pm \frac{2}{3}x$.

15. $\frac{y^2}{6^2} - \frac{x^2}{6^2} = 1$; $F(0, \pm 6\sqrt{2})$; $V(0, \pm 6)$; latus rectum, 12; asymptotes, $y = \pm x$.

17. $\frac{y^2}{2^2} - \frac{x^2}{10^2} = 1$; $F(0, \pm 2\sqrt{26})$; $V(0, \pm 2)$; latus rectum, 100; asymptotes, $y = \pm \frac{1}{5}x$.

19. $\frac{y^2}{9^2} - \frac{x^2}{(\frac{9}{8})^2} = 1$; $F(0, \pm\frac{9}{8}\sqrt{65})$; $V(0, \pm 9)$; latus rectum, $\frac{9}{32}$; asymptotes, $y = \pm 8x$.

21. $\frac{x^2}{36} - \frac{y^2}{64} = 1$. **23.** $\frac{x^2}{9} - \frac{y^2}{16} = 1$. **25.** $\frac{x^2}{49} - \frac{y^2}{49} = 1$.

27. $\frac{x^2}{5} - \frac{11y^2}{20} = 1$. **29.** $\frac{y^2}{16} - \frac{x^2}{9} = 1$. **31.** $\frac{y^2}{16} - \frac{x^2}{21} = 1$.

Page 134

7. Positive x-axis through F, perpendicular to directrix; origin $\frac{p}{e^2 - 1}$ units from directrix.

Page 141

3. $e = \frac{1}{2}$; directrix, $r \cos\theta = 8$.
5. $e = \frac{4}{3}$; directrix, $r \cos\theta = \frac{5}{4}$.
7. $e = \frac{5}{3}$; directrix, $r \cos\theta = -\frac{12}{5}$.
9. $e = \frac{3}{4}$; directrix, $r \cos\theta = -\frac{8}{3}$.

Pages 143, 144

1. $A(4, 3)$, $B(-3, -3)$, $C(-2, -5)$.
3. $x'^2 + y'^2 = 25$. **5.** $x'^2 + y'^2 = r^2$.
7. $y'^2 = 6x'$. **9.** $4x'^2 + 9y'^2 = 16$.
11. $16x'^2 + 9y'^2 = 49$. **13.** $9x'^2 - 4y'^2 = 36$.

Pages 146, 147

1. $V(2, 0)$; $F(2, \frac{1}{4})$; axis, $x = 2$; directrix, $y = -\frac{1}{4}$.

3. $V(-4, -6)$; $F(-1, -6)$; axis, $y = -6$; directrix, $x = -7$.

5. Vertices, $(1, 1)$, $(-9, 1)$; foci, $(-4 \pm \frac{5}{2}\sqrt{3}, 1)$; axes, $y = 1$, $x = -4$; directrices, $x = -4 \pm \frac{10}{3}\sqrt{3}$.

7. Vertices, $(1, -5)$, $(1, 13)$; foci, $(1, 4 \pm 6\sqrt{2})$; axes, $x = 1$, $y = 4$; directrices, $y = 4 \pm \frac{27}{4}\sqrt{2}$.

9. Vertices, $(0, 0)$, $(-20, 0)$; foci, $(-10 \pm 10\sqrt{2}, 0)$; axes, $y = 0$, $x = -10$; directrices, $x = -10 \pm 5\sqrt{2}$; asymptotes, $y = \pm(x + 10)$.

11. Vertices, $(1, -1)$, $(1, -9)$; foci, $(1, -5 \pm \frac{4}{3}\sqrt{10})$; axes, $x = 1$, $y = -5$; directrices, $y = -5 \pm \frac{6}{5}\sqrt{10}$; asymptotes, $y + 5 = \pm 3(x - 1)$.

13. $(y - 4)^2 = \pm 8(x - 3)$. **15.** $(x + 4)^2 = -8(y - 2)$.

17. $\frac{(x+2)^2}{9} + \frac{(y-2)^2}{36} = 1$. **19.** $\frac{(x-4)^2}{9} + \frac{(y - \frac{11}{4})^2}{\frac{169}{16}} = 1$.

21. $\frac{(x-2)^2}{9} - \frac{(y+4)^2}{16} = 1$. **23.** $\frac{(y-4)^2}{16} - \frac{(x-6)^2}{9} = 1$.

25. $V\left(-\frac{B}{2A}, \frac{4AC - B^2}{4A}\right)$, $F\left(-\frac{B}{2A}, \frac{4AC - B^2 + 1}{4A}\right)$.

Pages 149, 150

1. $A(6, 0)$; $B(0, -6)$; $C(3, 3)$.
3. $A(1 + \sqrt{3}, 1 - \sqrt{3})$; $B(2\sqrt{3}, 2)$; $C(-1, \sqrt{3})$.
5. $A(-3, -3\sqrt{3})$; $B(-2 + 2\sqrt{3}, -2 - 2\sqrt{3})$; $C(2 - 2\sqrt{3}, 2 + 2\sqrt{3})$.
7. $x'^2 - y'^2 = 36$, rectangular hyperbola.
9. $x'^2 + y'^2 = a^2$, circle.
11. $x'^2 = 25$, pair of straight lines.
13. $x'^2 = -2py'$, parabola.
15. $b^2y'^2 - a^2x'^2 = a^2b^2$, hyperbola.

Pages 152, 153

1. If A is $(7, 0)$ and BC is $x = -3$, the equation is $y^2 = 28x$, parabola.

3. If the points are $(\pm a, 0)$, and k is the constant, the equation is $x^2 + y^2 = \frac{k}{2} - a^2$, circle.

5. If the points are $(\pm a, 0)$, and k is the constant, the equation is $[(x + a)^2 + y^2][(x - a)^2 + y^2] = k$.

7. $\frac{x^2}{12} + \frac{(y+2)^2}{16} = 1$, ellipse.

9. $\frac{(y-8)^2}{16} - \frac{x^2}{48} = 1$, hyperbola.

11. $(x + y)^2 = 8(x - y + 2)$, parabola.
13. $7x^2 + 2xy + 7y^2 - 8x + 8y - 16 = 0$, ellipse.
15. $x^2 - 4xy + y^2 + 16x - 16y + 32 = 0$, hyperbola.
17. $(1 - e^2)x^2 + y^2 + 2e^2px - e^2p^2 = 0$, conic section.
19. $(x - y)^2 = -2(5\sqrt{2} + 20)(x + y) + (5\sqrt{2} + 20)^2$, parabola.

21. The left branch of the hyperbola $x^2 - \frac{y^2}{15} = 1$, and the right branch of the hyperbola $\frac{x^2}{4} - \frac{y^2}{12} = 1$.

23. $9x^2 - 16y^2 - 27x - 20y - 236 = 0$, hyperbola.

Page 156

1. Intercepts, 0, 0; symmetrical to origin.

3. Intercepts, ± 5, ± 5; symmetrical to origin and both axes.

5. Intercepts, $x = \pm 2$, no y-intercept; symmetrical to origin and both axes.

7. Intercepts, 0, 0; symmetrical to x-axis.

9. x-intercepts, 0, -1; y-intercept, 0; symmetrical to x-axis.

11. Intercepts, 0, 0; symmetrical to x-axis.

13. Intercepts, 0, 0; symmetrical to origin.

15. x-intercepts, -1, 0, 1, y-intercept, 0; symmetrical to origin.

17. Intercepts, $\pm\sqrt{7}$, $\pm\sqrt{7}$; symmetrical to origin.

Page 159

1. No excluded values. **3.** Negative values of x are excluded.

5. Curve bounded by $x = -1$, $x = 9$, $y = -5$, $y = 5$.

7. Curve bounded by $x = -6$, $x = 6$, $y = -4$, $y = 4$.

9. Values of x between -2 and 2 are excluded.

11. Values of x between -4 and 2 are excluded.

13. Negative values of x are excluded, and values of $y < -\frac{5}{4}$.

15. Curve bounded by $x = -1$, $x = 3$, $y = -4$, $y = 0$.

17. No excluded values.

19. Curve bounded by $x = -4$, $x = 4$, $y = -\sqrt{2}$, $y = \sqrt{2}$.

21. All values of x such that $x \leqq -2$, and all such that $1 \leqq x < 5$, are excluded; all values of y between $-\frac{1}{3}(\sqrt{7} + 2)$ and $-\frac{1}{3}(\sqrt{7} - 2)$, and all between $\frac{1}{3}(\sqrt{7} - 2)$ and $\frac{1}{3}(\sqrt{7} + 2)$, are excluded.

Page 167

1. (3, 1). **3.** $(2 + \frac{5}{2}\sqrt{2}, -3 - \frac{5}{2}\sqrt{2})$, $(2 - \frac{5}{2}\sqrt{2}, -3 + \frac{5}{2}\sqrt{2})$.

5. $(8, -3)$, $(-6, 4)$. **7.** $(2, 2\sqrt{3})$, $(2, -2\sqrt{3})$.

9. (2, 4), (4, 2), $(-2, -4)$, $(-4, -2)$.

11. (1, 6), (6, 1), $(-1, -6)$, $(-6, -1)$.

13. $(\frac{5}{2}\sqrt{2}, \frac{5}{2}\sqrt{2})$, $(-\frac{5}{2}\sqrt{2}, -\frac{5}{2}\sqrt{2})$.

Page 180

25. $x^2 + y^2 - 10x = 0$, $x^2 + y^2 - 10y = 0$, $(x^2 + y^2)^3 = 100(x^2 - y^2)^2$, $x = 10$.

27. $(x^2 + y^2 + bx)^2 = a^2(x^2 + y^2)$, the origin excepted if $a > b$, $x(x^2 + y^2) - 2ay^2 = 0$.

29. $(r^2 - 2r\cos\theta)^2 = 16$. **31.** $(r + 4\sin\theta)^2 = 4$.

Pages 180, 182

1. $r = \frac{ep}{1 + e\sin\theta}$. **3.** $r^2 = 2a^2\cos 2\theta$.

5. $r = 2a(1 + \cos\theta)$. **7.** $r^2 = a^2(\cos^6\theta + \sin^6\theta)$.

Page 186

1. $\left(4, \frac{\pi}{3}\right)$, $\left(4, -\frac{\pi}{3}\right)$. **3.** $\left(2\sqrt{3}, \frac{\pi}{3}\right)$, $\left(2\sqrt{3}, \frac{2\pi}{3}\right)$.

5. $\left(1, \frac{\pi}{6}\right)$, $\left(1, \frac{5\pi}{6}\right)$, $\left(2, \frac{\pi}{2}\right)$, and the pole.

7. (0.219, 231°), (0.219, $-51°$), (1, 180°), (1, 0°), (0.5, 210°), (0.5, $-30°$), and the pole.

9. $(\sqrt{2}, -30°)$, $(\sqrt{2}, 30°)$, $(\sqrt{2}, 150°)$, $(\sqrt{2}, 210°)$, and the pole.

11. (0.92, 262°), (0.92, 278°), and the pole.

Page 189

1. $y = \frac{3}{2}x + 4$. **3.** $x^2 + (y - 2)^2 = 36$. **5.** $y^2 = x^3$.

7. $\frac{x^2}{25} - \frac{y^2}{16} = 1$. **9.** $x^3 + y^3 = 3axy$. **11.** $\alpha = 45°$

13. Yes, for the example given, but not for some other examples.

Pages 194, 195

1. $x = \frac{3at}{1 + t^3}$, $y = \frac{3at^2}{1 + t^3}$.

3. $x = a\theta - l\sin\theta$, $y = a - l\cos\theta$.

5. $r = 2a(1 - \cos\theta')$, where (r, θ') are polar coördinates of P.

9. $x = a\cos^3\theta$, $y = a\sin^3\theta$.

11. $x = (a + b)\cos\theta - l\cos\frac{a + b}{b}\theta$,

$y = (a + b)\sin\theta - l\sin\frac{a + b}{b}\theta$.

Pages 201, 202

1. $x_1x = p(y + y_1)$. **3.** $x_1x - y_1y = a^2$.

5. $x_1x + y_1y = a(x + x_1)$. **7.** $ax_1x + b(x_1y + y_1x) + cy_1y = 0$.

11. $(x_1^2 - ay_1)(x - x_1) + (y_1^2 - ax_1)(y - y_1) = 0$.

13. $(3x_1^2 + y_1^2 + 2ax_1)(x - x_1) + 2y_1(x_1 - a)(y - y_1) = 0$.

15. $2x + y + 2 = 0$. **17.** $2x + 7y = 53$.

19. $3x + 4y = 26$. **21.** $2x + \sqrt{5}y = 8$.

23. $x + 2y - 18 = 0$.

27. $3x + 2\sqrt{3}y = 24$, $11x - \frac{10}{3}\sqrt{3}y = 56$.

Pages 204, 205

1. $4x - 3y = 0$, $\frac{20}{3}$, 5, $-\frac{16}{3}$, 3.

3. $3x + 2y - 51 = 0$, $6\sqrt{13}$, $4\sqrt{13}$, 18, -8.

5. $8x - 3y - 18 = 0$, $\frac{2}{3}\sqrt{73}$, $\frac{1}{4}\sqrt{73}$, $-\frac{16}{3}$, $\frac{3}{4}$.

7. $16x + 5y - 170 = 0$, $\frac{2}{5}\sqrt{281}$, $\frac{1}{8}\sqrt{281}$, $\frac{32}{5}$, $-\frac{5}{8}$.

9. $5x - 6y + 15 = 0$, $\frac{5}{6}\sqrt{61}$, $\sqrt{61}$, $-\frac{25}{6}$, 6.

11. $4x + y + 10 = 0$, $2\sqrt{17}$, $\frac{1}{2}\sqrt{17}$, 8, $-\frac{1}{2}$.

13. $x - 3y - 28 = 0$, $\frac{8}{3}\sqrt{10}$, $8\sqrt{10}$, $\frac{8}{3}$, -24.

15. $2x - y - 3a = 0$, $a\sqrt{5}$, $\frac{a}{2}\sqrt{5}$, $-2a$, $\frac{a}{2}$.

17. $x + 6 = 0$, length of tangent and subnormal do not exist, length of normal $= 15$, subnormal $= 0$.

Page 207

1. $3x + 4y - 25 = 0$, $3x + 4y + 25 = 0$.

3. $8x + 4y - 13 = 0$.

5. $x - y - 2\sqrt{37} = 0$, $x - y + 2\sqrt{37} = 0$.

7. $2x - y + 9\sqrt{3} = 0$, $2x - y - 9\sqrt{3} = 0$

Page 213

5. $e = \frac{1}{3}$; $y = \pm\dfrac{2\sqrt{2}}{3}x$.

Page 215

3. $y = 2$. **5.** $y = -8$. **7.** $x = 8$

Pages 218, 219

1. Polar of A, $3x + 3y = 15$; of B, $6x - 5y = 15$;
of C, $4x + 5y = -25$; of D, $4x = -25$;
of E, $18x - 50y = -225$.

3. Polar of A, $3x - 5y = 15$; of B, $6x + 10y = 15$;
of C, $6x + 5\sqrt{3}y = -15$; of D, $4x = -25$;
of E, $-18x + 50y = 225$.

Pages 221, 222

1. $(18, 6)$. **3.** $(\frac{25}{4}, \frac{9}{2})$. **5.** $(\frac{100}{3}, -\frac{200}{3})$.

7. $(-2, 8)$. **9.** $(-21, 28)$. **11.** $(-8, 0)$; $(-4, 4)$.

15. Pole of $y = k$ is $\left(0, \dfrac{b^2}{k}\right)$; of $x = k$ is $\left(\dfrac{a^2}{k}, 0\right)$.

Page 227

9. $(18, \frac{9}{4})$.

Pages 232, 233

1. $h = -2$, $k = 1$; $x'^2 + y'^2 = 8$.

3. $h = 1$, $k = 1$; $x'^2 - 4y'^2 + 6 = 0$.

5. $h = 0$, $k = -1$; $3x'^2 + 4x'y' = 2$.

7. $h = 4$, $k = 11$; $8x'^2 - 6x'y' + y'^2 + 16 = 0$.

9. $h = 3$, $k = -1$; $2x'^2 - 5x'y' + 2y'^2 - 35 = 0$.

11. $x'^2 + 4y'^2 = 4$; center, $(1, -\frac{1}{2})$; foci, $(1 \pm \sqrt{3}, -\frac{1}{2})$.

13. $9x'^2 - y'^2 = 9$; transverse axis on $y = 1$;
directrices, $x = \pm\dfrac{\sqrt{10}}{10}$; asymptotes, $9x^2 - (y - 1)^2 = 0$.

15. $9x'^2 + 4y'^2 = 36$; lines of axes, $x - 2 = 0$, $y + 1 = 0$;
directrices, $y = -1 \pm \frac{9}{5}\sqrt{5}$.

Pages 236, 237

1. $x = \dfrac{x' - y'}{\sqrt{2}}$, $y = \dfrac{x' + y'}{\sqrt{2}}$; $3x'^2 - y'^2 + \sqrt{2}(x' - y') = 4$.

3. $x = \dfrac{3x' - 4y'}{5}$, $y = \dfrac{4x' + 3y'}{5}$;
$-12x'^2 + 13y'^2 + 3x' - 4y' = 0$.

5. $x = \dfrac{x' - y'}{\sqrt{2}}$, $y = \dfrac{x' + y'}{\sqrt{2}}$; $2x'^2 + 8\sqrt{2}y' = 4$.

7. $x = \dfrac{3x' - 4y'}{5}$, $y = \dfrac{4x' + 3y'}{5}$;
$20x'^2 - 5y'^2 - 6x' - 17y' = 1$.

9. $x = \dfrac{3x' - y'}{\sqrt{10}}$, $y = \dfrac{x' + 3y'}{\sqrt{10}}$; $5x'^2 + 45y'^2 = 0$.

11. $\dfrac{x'^2}{1^2} + \dfrac{y'^2}{2^2} = 1$; center, $(0, 0)$; foci, $(-\sqrt{\tfrac{3}{2}}, \sqrt{\tfrac{3}{2}})$, $(\sqrt{\tfrac{3}{2}}, -\sqrt{\tfrac{3}{2}})$.

13. $y'^2 - x'^2 = 1$; directrices, $3x - 4y \pm \tfrac{5}{2}\sqrt{2} = 0$.

15. $\dfrac{x'^2}{3} - \dfrac{y'^2}{10} = 1$; center, $(0, 0)$;
vertices, $(3\sqrt{\tfrac{3}{13}}, 2\sqrt{\tfrac{3}{13}})$, $(-3\sqrt{\tfrac{3}{13}}, -2\sqrt{\tfrac{3}{13}})$.

Pages 242, 243

1. $4x''^2 + y''^2 - 4 = 0$.
3. $4x''^2 - y''^2 - 4 = 0$.
5. $y''^2 + 4x'' = 0$.
7. $x''^2 - 4y''^2 = 0$.
9. $2x''^2 + y''^2 + 4 = 0$.
11. $x''^2 - y''^2 + 1 = 0$.
13. $y''^2 - 4 = 0$.
15. $16x''^2 + y''^2 = 0$.

Pages 247, 248

1. Point ellipse.
3. Ellipse; $e = \dfrac{\sqrt{8}}{3}$.
5. One line.
7. Hyperbola; $e = \tfrac{2}{3}\sqrt{3}$.
9. Intersecting lines.
11. $a + c = 0$.

13. $\dfrac{2}{a + c}\sqrt{\dfrac{-\Delta}{a + c}}$.

17. $e = \sqrt{1 - \dfrac{r_1}{r_2}}$, where r_1 and r_2 are roots of the equation
$r^2 - (a + c)r - (b^2 - ac) = 0$, such that $(r_1 - r_2)\,\Delta > 0$.

Page 253

1. $x = \tfrac{4}{3}\sqrt{3}$, $y = -\tfrac{2}{3}\sqrt{3}$; $x = -\tfrac{4}{3}\sqrt{3}$, $y = \tfrac{2}{3}\sqrt{3}$;
$x = -\tfrac{5}{2}$, $y = -\tfrac{3}{2}$. All are intersections.

3. $x = 0$, $y = 0$; $x = \tfrac{4}{5}$, $y = \tfrac{4}{5}$. Both are intersections.

5. $x^2 + y^2 + 4x + 4y - 17 = 0$.

7. $x^2 - xy + y^2 = 4$.

9. $x^2 - y^2 + 2x + 1 = 0$.

11. $x^2 - 2xy + y^2 - 9x + 6y + 8 = 0$;
$x^2 + 2xy + y^2 - x + 2y = 0$.

Pages 259, 260

1. $y = 11.64 - 0.24x$.
3. $y = 11.64 - 0.24x$.

5. $y = \dfrac{19{,}822 - 976x + 19x^2}{1323} = 15.0 - 0.738x + 0.0144x^2$.

7. $y = \dfrac{18{,}974 - 963x + 19x^2}{1260} = 15.1 - 0.764x + 0.0151x^2$.

9. (*a*) $a = 10.76$; 10.76; 43.04; 96.84; 172.16.
(*b*) $a = 7.174$; 7.17; 28.70; 64.57; 114.78.

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1. (*a*) $a = 61.9$; $\sigma = 20.8$. (*b*) $a = 90.0$; $\sigma = 1.75$.
3. (*a*) $a = 60.7$; $\sigma = 20.2$. (*b*) $a = 79.7$; $\sigma = 3.20$.
5. (*a*) $a = 61.8$; $\sigma = 19.4$. (*b*) $a = 65.7$; $\sigma = 1.62$.

Pages 270–272

1. $y = \dfrac{1817 - 34x}{165} = 11.01 - 0.206x$.

3. $y = \dfrac{326{,}426 + 38{,}586x}{43{,}369} = 7.53 + 0.890x$.

5. $y = \dfrac{-3372 + 12{,}549x}{13{,}251} = -0.254 + 0.947x$.

7. $y = \dfrac{-0.338 + 0.0738x}{11} = -0.0307 + 0.00671x$.

11. $\bar{x} = 60$; $\bar{y} = 0.372$; $\sigma_x = 31.6$; $\sigma_y = 0.212$; $r = 0.9988$;
$y = -0.0307 + 0.00671x$.

Page 274

1. $d = 0.494 - 1.314t + 7.53t^2$.

3. $y = \dfrac{-125{,}296 + 4452x - 23x^2}{2688}$
$= -46.61 + 1.656x - 0.008557x^2$.

Pages 280, 281

1. $d = 7.399t^{1.969}$.
3. $i = 0.003475v^{1.1354}$.
5. $x = (0.08462)(2.820)^y$.

Page 285

7. $z = 0$; $x = 0$: $y = 0$.

Pages 288, 289

1. (a) $\sqrt{82}$; (b) 1; (c) $\sqrt{56}$; (d) $\sqrt{52}$.

3. $2(x_1 - x_2)x + 2(y_1 - y_2)y + 2(z_1 - z_2)z$
$= x_1^2 - x_2^2 + y_1^2 - y_2^2 + z_1^2 - z_2^2$.

5. $(x-2)^2 + (y-4)^2 + (z-6)^2 = 36$.

7. (a) $C(2, 3, -4)$, $R = 6$; (b) $C(-3, 4, 0)$, $R = 5$.

9. $x^2 + y^2 + z^2 - 46x - 22y + 14z + 40 = 0$.

Pages 292, 293

1. (a) $\dfrac{1}{\pm 3}, \dfrac{2}{\pm 3}, \dfrac{-2}{\pm 3}$; (b) $0, \dfrac{4}{\pm 5}, \dfrac{3}{\pm 5}$.

3. (a) $\dfrac{1}{\pm\sqrt{6}}, \dfrac{1}{\pm\sqrt{6}}, \dfrac{-2}{\pm\sqrt{6}}$; (b) $\dfrac{1}{\pm\sqrt{18}}, \dfrac{1}{\pm\sqrt{18}}, \dfrac{-4}{\pm\sqrt{18}}$.

5. (a) $\dfrac{1}{\pm 3}, \dfrac{2}{\pm 3}, \dfrac{-2}{\pm 3}$; (b) $\dfrac{3}{\pm 5}, 0, \dfrac{-4}{\pm 5}$.

7. (a) $\dfrac{1}{\pm\sqrt{2}}, 0, \dfrac{1}{\pm\sqrt{2}}$; (b) $\dfrac{1}{\pm\sqrt{3}}, \dfrac{1}{\pm\sqrt{3}}, \dfrac{1}{\pm\sqrt{3}}$.

9. (a) $l = \dfrac{1}{2}$, $m = \dfrac{1}{\sqrt{2}}$, $n = \pm\dfrac{1}{2}$; $\gamma = 60°$ or $120°$.

(b) $l = -\dfrac{1}{\sqrt{2}}$, $m = 0$, $n = \dfrac{\pm 1}{\sqrt{2}}$; $\gamma = 45°$ or $135°$.

11. (a) $l = 0$, $m = \dfrac{1}{\sqrt{2}}$, $n = \dfrac{1}{\sqrt{2}}$, $\alpha = 90°$.

(b) $l = \dfrac{\pm 1}{\sqrt{3}}$, $m = \dfrac{\pm 1}{\sqrt{3}}$, $n = \dfrac{\pm 1}{\sqrt{3}}$, $\alpha = \beta = \gamma = 54° 44'$ or $125° 16'$.

13. Of x-axis, $l = 1$, $m = 0$, $n = 0$.

15. $l = \dfrac{1}{\sqrt{2}}$, $m = 0$, $n = \dfrac{1}{\sqrt{2}}$.

Pages 295, 296

1. (a), (d), and (e); (b) and (f).

3. (a) and (b); (b) and (c); (b) and (d); (a) and (e); (c) and (e); (d) and (e).

5. $\cos^{-1}\dfrac{\pm 2}{15}$. **7.** $\cos^{-1}\dfrac{\pm 5}{3\sqrt{3}}$. **9.** $\cos^{-1}\dfrac{\pm 11}{15}$.

11. $\cos^{-1}\dfrac{\pm 2}{\sqrt{6}}$. **13.** $l = \dfrac{1}{\pm\sqrt{2}}$, $m = \dfrac{-1}{\pm\sqrt{2}}$, $n = 0$.

15. $\dfrac{\sqrt{6}}{2}$. **23.** $\dfrac{x_1 + x_2}{2}, \dfrac{y_1 + y_2}{2}, \dfrac{z_1 + z_2}{2}$.

Page 300

1. $\frac{1}{3}x - \frac{2}{3}y + \frac{2}{3}z - 1 = 0$; 1; $\frac{1}{3}, -\frac{2}{3}, \frac{2}{3}$.

3. $-\frac{3}{5}x + \frac{4}{5}z - 1 = 0$; 1; $-\frac{3}{5}, 0, \frac{4}{5}$.

5. $\frac{8}{17}x + \frac{15}{17}y = 0$; 0; $\frac{8}{17}, \frac{15}{17}, 0$.

7. $-\frac{7}{6}$; same side as the origin.

9. $\frac{4}{5}$; opposite side from the origin.

11. 0; point is on the plane.

13. $x - 2y - 3z \pm 4\sqrt{14} = 0$. **15.** $\frac{24}{5}$.

17. $\dfrac{3x + 4y + 12z - 13}{13} = \pm\dfrac{7x - 24y + 50}{25}$.

Pages 304, 305

1. $\cos^{-1} \pm \frac{3}{5}$. **3.** $90°$. **5.** Parallel.

7. $\cos^{-1}\dfrac{\pm 19}{3\sqrt{82}}$. **9.** $3x + 4y + 12z = 66$.

11. $x + y - 2z - 3 = 0$. **13.** $2x - 4y + z = 4$.

15. $x + y = 4$. **17.** $y = 0$. **19.** $z = 1$.

21. $y + z = 0$. **23.** $\dfrac{x}{a} + \dfrac{y}{b} = 1$.

Pages 310, 311

1. $\dfrac{x-3}{2} = \dfrac{y-1}{3} = \dfrac{z-2}{-2}$. **3.** $y = 0$, $z = 4$.

5. $\dfrac{x-2}{2} = \dfrac{y-1}{-2} = \dfrac{z-2}{-2}$. **7.** $y = 0$, $z = 4$.

9. $x = y = \dfrac{z}{2}$. **11.** $x = \dfrac{y-1}{3} = \dfrac{z-2}{2}$.

13. $x = 3 + 2t$, $y = 1 + 3t$, $z = 2 - 2t$;
$x = 3t$, $y = 0$, $z = -t$.

15. $x = x_0 + ls$, $y = y_0 + ms$, $z = z_0 + ns$.

17. $\dfrac{1}{9}, \dfrac{-4}{9}, \dfrac{8}{9}, \dfrac{x}{1} = \dfrac{y-1}{-4} = \dfrac{z+3}{8}$.

19. $z = 0, 4x + y = 1$; $x = 0, 2y + z = -1$; $y = 0, 8x - z = 3$.

Pages 311, 313

1. 4, − 4, 2. **3.** $(-1, -2, 0), \left(\frac{1}{3}, 0, \frac{-2}{3}\right), \left(0, \frac{-1}{2}, \frac{-1}{2}\right)$.

5. Below; above; on; above. **9.** $5x - 7y - 4z = 9$.

11. $\frac{23}{27}$. **13.** $k = 1$. **15.** $3x - y + 2z + 1 = 0$.

17. $\frac{\sqrt{595}}{17}$. **19.** $\frac{\sqrt{70}}{2}$. **23.** $\begin{vmatrix} A_1 & B_1 & C_1 \\ A_2 & B_2 & C_2 \\ A_3 & B_3 & C_3 \end{vmatrix} = 0$.

Pages 326, 327

1. Projection on xy-plane is a parabola; on xz-plane, a parabola; on yz-plane, a straight line.

3. Projection on xy-plane is a circle; on xz-plane, a part of two lines parallel to x-axis; on yz-plane a part of two lines parallel to y-axis.

5. If no two of a, b, c are equal, the projection on each coördinate plane is an ellipse.

Pages 330, 331

1. Ellipsoid of revolution. **3.** Hyperboloid of one sheet.

5. Hyperbolic paraboloid.

9. $x = \frac{x' + y' + \sqrt{2}z'}{2}, \quad y = \frac{x' + y' - \sqrt{2}z'}{2}, \quad z = \frac{x' - y'}{\sqrt{2}}$.

Pages 332, 333

1. Right circular cylinder; $x^2 + y^2 = 2ax$. **3.** Sphere.

5. Line parallel to z-axis. **7.** Circle parallel to xy-plane.

9. Sphere. **11.** Hyperboloid of two sheets.

13. Circle. **15.** Straight line. **17.** Conical helix.

19. $r^2 + z^2 = a^2$; $\rho = a$.

21. $z = r^2$; $\rho^2 \sin\phi \tan\phi = \rho$.

23. $\frac{r^2}{a^2} + \frac{z^2}{c^2} = 1, \quad \rho^2\left(\frac{\sin^2\phi}{a^2} + \frac{\cos^2\phi}{c^2}\right) = 1$.